MATHEMATICS

PART II

A Textbook for Secondary Schools

MATHEMATICS

PART II

A Textbook for Secondary Schools

Prescribed by the Central Board of Secondary Education

National Council of Educational Research and Training

Rs 3.45

PUBLISHED AT THE PUBLICATION DEPARTMENT BY V. R. DRAVID, SECRETARY, NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING, SRI AUROBINDO MARG, NEW DELHI 110016 AND PRINTED AT RAJENDRA RAVINDRA PRINTERS (PVT) LTD, RAMNAGAR, NEW DELHI 110055

FOREWORD

The 10+2 pattern of education is being introduced in the country. The Central Board of Secondary Education, in its network of 1000 and odd schools, introduced this pattern last year. The system, which carried the imprint of approval of numerous bodies concerned with various aspects of education, has the potentiality of equipping a student to face successfully the challenges of modern technological society. Naturally, mathematics has come to occupy a central position in the set-up. There is no subject in which mathematics does not enter in some measure. It is also difficult to imagine any aspect of human activity in which mathematics has no contribution to make.

While the NCERT's own syllabus based on country-wide consultations, finally accepted by distinguished academicians, is being implemented, the Council, on the request of the Central Board of Secondary Education, prepared a book for Class IX last year. The present one is a sequel to it. The philosophy behind this syllabus is to provide some mathematical literacy to the student for whom the study of mathematics will be terminal with Class X as also to the student who will go on with mathematics.

The first draft of this book was developed by Shri G.S. Baderia, Dr. B. Deokinandan, Shri R. C. Saxena, Dr. Ram Autar, Shri G. D. Dhall, Shri Ishwar Chandra and Shri Mahendra Shanker under the overall guidance of Dr. Manmohan Singh Arora, all of the Department of Education in Science and Mathematics. This draft was thoroughly discussed by experienced teachers and subject-specialists at the Workshop held in Udaipur from 29 September to 8 October 1975 to ensure that the needs of the student are met in the best possible manner. The final writing and content-editing of the book was done by Dr. Manmohan Singh Arora. In this task, necessary assistance was provided by Dr. B. Deokinandan, Dr. Ram Autar, Shri G.D. Dhall, Dr. S.K. Singh Gautam, Dr. T.J.S. Mehrok and Shri Mahendra Shanker. Answers were supplied by Dr. S.K. Singh Gautam and Dr. T.J.S. Mehrok.

The text was rendered into Hindi by Shri S.C. Saksena of the Central Hindi Directorate. The content-editing of the Hindi version was done by Dr. B. Deokinandan, Dr. S.K. Singh Gautam and Shri Mahendra Shanker, with the co-operation of Dr. T.J.S. Mehrok, Shri Ishwar Chandra and Shri R.D. Sharma. I owe grateful thanks to all of them. I am also grateful to the staff of the Publica-

tion Department who took pains in bringing out the book.

The Council would welcome the reactions of the users to incorporate improvements in the future books to be brought out according to the NCERT's syllabus

New Delhi
March 1976

RAIS AHMED
Director
National Council of Educational
Research and Training

PREFACE

It has been the constant endeavour of the Central Board of Secondary Education to update and improve their syllabi in various subjects in order to provide the best possible training to the students. Precisely for this reason, the Board changed over to the 10+2 pattern just a year ago. While the earlier system had outlived whatever utility it had, the latter had received overwhelming support from numerous academic and even non-academic forums over the past six decades The 10+2 system, thus, has a very sound base and aims at re-vitalising and rationalising school curricula by recasting them on more scientific lines so that these can meet efficiently the needs and urges of an adolescent. Expert Committees were accordingly set up who have laid down appropriate, forward-looking syllabi for various subjects to be in phase with the exacting demands of the modern society.

The Board then embarked on the next vital step of preparation of suitable textbooks and requested the experts at the National Council of Educational Research and Training to devise suitable textual material. Textbook occupies a key position in the whole scheme of innovations and improving education. It makes the syllabus pulsate with life. I am very happy to see this second part of the book on Mathematics. The choice of the material as also its presentation are highly commendable. These can be expected to generate interest and initiative in this important subject even in an average student which will prove to be an asset to him irrespective of the fact whether his formal education terminates with Class X or he goes in for higher studies. For this the credit goes to the vision and diligence of Professor Manmohan Singh Arora and his able team of collaborators in the Department of Education in Science and Mathematics of the National Council of Educational Research and Training, New Delhi. I have been in contact with Professor Manmohan Singh Arora all through the preparation of this book. The materials developed by Professor Arora and his team have been examined carefully by our Expert Committee on Mathematics and we are glad that this book has also shaped as well as its predecessor for which we have received some very encouraging comments from several members of the teaching community.

The National Council of Educational Research and Training and the Central Board of Secondary Education will be glad to consider any suggestions for enhancing the usefulness of this book from all those who go through it.

New Delhi
March 1976

GOVERDHAN LAL BAKHSHI
Chairman
Central Board of Secondary Education
New Delhi

ACKNOWLEDGEMENTS

The National Council of Educational Research and Training is grateful to the following who made a significant contribution towards the development of this textbook in a Workshop held for the purpose at the State Institute of Science Education, Udaipur, in October, 1975 :

1. Dr. Manmohan Singh Arora
   National Council of Educational Research and Training, New Delhi
2. Dr. Ram Autar
   National Council of Educational Research and Training, New Delhi
3. Shri N.B. Badrinarayan
   Regional College of Education, Mysore
4. Dr C.P. Bajaj
   Rajdhani College, New Delhi
5. Dr. G.L. Bakhshi
   Central Board of Secondary Education, New Delhi
6. Shri V.S. Bhandari
   State Institute of Science Education, Udaipur
7. Dr. M.P. Bhasin
   Ramgarhia College of Education, Phagwara
8. Shri D.P. Bhatnagar
   State Institute of Science Education, Udaipur
9. Dr. B. Chaudhuri
   Jadavpur University, Calcutta
10. Dr. K.P. Chinda
    Hindu College, Delhi
11. Dr. S.D. Chopra
    Kurukshetra University, Kurukshetra
12. Shri Bhagwan Das
    State Institute of Education, Delhi
13. Shri G.D. Dhall
    National Council of Educational Research and Training, New Delhi
14. Shri G.S. Dhillon
    Guru Harkrishan Public School, New Delhi

15. Dr. M.L. Gogna
Kurukshetra University, Kurukshetra
16. Dr. N.K Govil
Indian Institute of Technology, New Delhi
17. Dr M.C. Gupta
University of Rajasthan, Jaipur
18 Dr. S. Izhar Husain
Aligarh Muslim University, Aligarh
19. Shri B L. Jain
Higher Secondary School, Vallabhnagar, Udaipur
20. Km. Shashi Jain
Punjab School Education Board, Chandigarh
21. Dr. Ravinder Kumar
Ramjas College, Delhi
22. Smt. Kanchan Manaktala
Gargi College, New Delhi
23. Shri B. Naidu
Government Junior College, Hyderabad
24. Shri Shanti Narayan
University of Delhi, Delhi
25. Shri K.S. Pandey
Vidya Bhawan Higher Secondary School, Udaipur
26. Dr. I.B.S. Passi
Kurukshetra University, Kurukshetra
27. Shri A.C. Patel
State Institute of Education, Ahmedabad
28. Dr. G.C. Patni
University of Rajasthan, Jaipur
29. Shri K.N. Nagesha Rao
The Rajkumar College, Rajkot
30. Shri G.S. Rathore
Birla Public School, Pilani
31. Shri V. Seshan
Atomic Energy Central School, Bombay
32. Shri B L. Seth
Government Higher Secondary School, Rajsamand (Rajasthan)
33. Shri Mahendra Shanker
National Council of Educational Research and Training, New Delhi

34. Shri B.P. Sharma
Government Secondary School, Bharu (Rajasthan)
35. Dr. B D. Sharma
University of Delhi, Delhi
36. Dr. Satish Shirali
Panjab University, Chandigarh
37. Shri Balbir Singh
Rajputana Rifles Heroes Memorial Higher Secondary School, Delhi Cantt.
38. Dr. B.V. Singhbal
Tata Institute of Fundamental Research, Bombay
39. Dr. D.K. Sinha
Jadavpur University, Calcutta
40. Dr. A.M. Vaidya
Gujarat University, Ahmedabad
41. Smt. Anil Wasan
Ramjas Girls School, New Delhi
42. Shri S.K. Wasan
Ramjas College, Delhi

The Council is also grateful to Dr. (Km.) A.K. Chilana, University of Delhi, for her valuable suggestions on various parts of the manuscript.

CONTENTS

CHAPTER XVIII

LINEAR PROGRAMMING—AN INTRODUCTION

18.1. Introduction

In Chapter IV, we studied some real-life problems leading to two or more simultaneous linear equations and inequations*. In this chapter, we shall see how we can use our knowledge of simultaneous linear equations and inequations in solving problems that arise in trade, commerce, industry and military operations, etc. We will, however, for the sake of simplicity, consider only simpler versions of some real-life problems from these fields. Let us, for instance, consider the following example .

Example 1 : A furniture dealer deals in only two items (simplified version of a real-life situation) tables and chairs. He has Rs 5000.00 to invest and a space to store at most 60 pieces. A table costs him Rs 250 00 and a chair Rs 50 00. He can sell a table at a profit of Rs 50.00 and a chair at a profit of Rs 15 00.

Assuming he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit ?

In this example, we observe that there are two types of activities—the dealer can invest his money in purchasing tables or chairs or combinations thereof. Furthermore, he would earn different profits by following different investment strategies.

Also, there are certain overriding conditions or constraints, namely, his investment is limited (to a maximum of Rs 5000.00) and so is his storage space (to a maximum of 60 pieces).

The number each of tables and chairs is, therefore, a **variable**. Each is necessarily non-negative. (Why ?) The profit of the dealer is a function of both these variables. The dealer, of course, would like to invest in such a way so as to maximize his profit.

The above is a typical problem from among the class of problems, called **optimization problems,** that deal with the allocation of limited resources, under certain overriding conditions or constraints, to obtain the best possible or optimal results in meeting the given objectives.

***The reader is advised to review Chapter IV.**

18.2. Mathematical Formulation of the Problem

Let us read the problem given in Section 18 1 carefully and formulate it mathematically. We note that

Maximum possible investment :		Rs 5000.00
Maximum storage space :		60 pieces of furniture
Cost of a		
	table :	Rs 250.00
	chair .	Rs 50.00
Possible profit on a		
	table :	Rs 50.00
	chair :	Rs 15.00

Let us denote by x, the number of tables and by y, the number of chairs that the dealer buys. Of course,

$$x \geqslant 0 \qquad (1)$$
$$y \geqslant 0 \qquad (2)$$

The dealer is constrained by the maximum amount he can invest and by the maximum number of pieces he can store. Stated mathematically,

$$250x+50y \leqslant 5000$$

or $$5x+ \quad y \leqslant 100 \qquad (3)$$

and $$x+ \quad y \leqslant 60 \qquad (4)$$

The dealer wants to invest in such a way so as to maximize his profit P, which stated as a function of x and y is

$$P = 50x+15y \qquad (5)$$

Mathematically, therefore, the problem now reduces to

Maximize $\quad P = 50x+15y$

subject to the constraints

$$5x+y \leqslant 100$$
$$x+y \leqslant 60$$
$$x \geqslant 0$$

and $$y \geqslant 0$$

The method of maximizing (or minimizing) a linear function of several variables (called OBJECTIVE FUNCTION) subject to the condition that the variables are non-negative and satisfy a set of linear equations and/or inequations (called LINEAR CONSTRAINTS) is given the name LINEAR PROGRAMMING The term **linear** implies that all the mathematical relations used in the problem are **linear relations**, while the term **programming** refers to the method of determining a particular **programme** or **plan** of action The two together have the technical meaning stated above.

18.3. The Graphical Method of Solving Linear Programming Problems

Let us refer to the problem given in Section 18.1 again and graph the constraints stated as inequations (1), (2), (3) and (4). (See Fig. 18.1) The shaded region consists of points

in the intersection of all the closed (why ?) half-planes (why ?) represented by the four constraints Each point in this region represents a feasible choice open to the dealer for

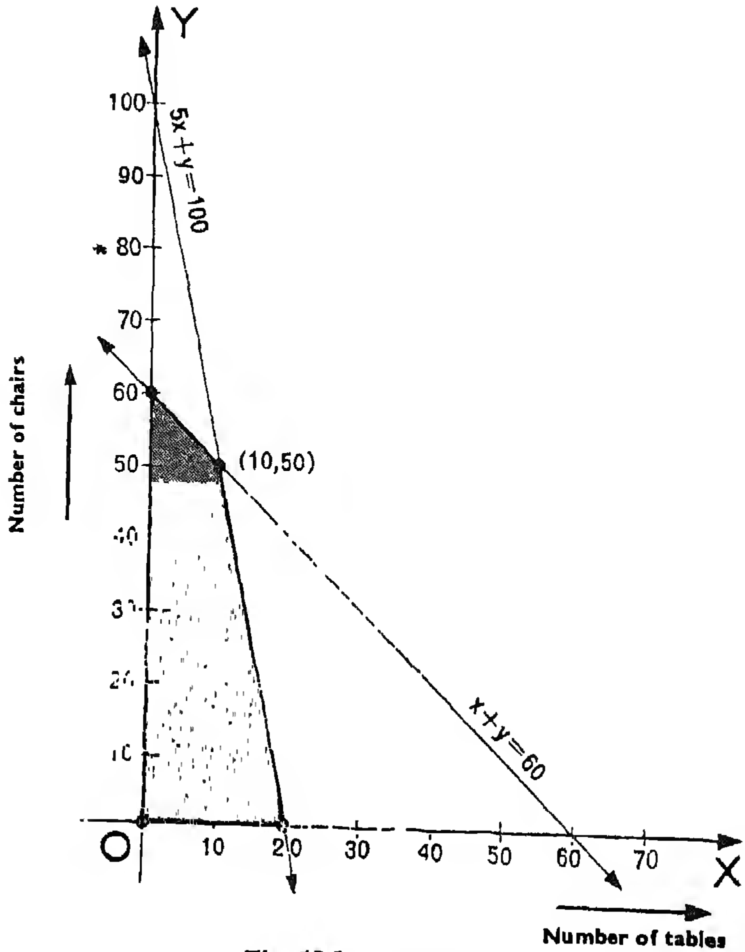


Fig. 18.1. Feasible region

investing in tables and chairs. The region, therefore, is called a **feasible region***. Every point of this region is called a **feasible solution** to the linear programming problem.

It is a matter of commonsense that the dealer would seek only that (those) feasible solution(s) which would maximize his profit, namely, the objective function stated as equation (5). What we could, therefore, do is make a table of P-values for different feasible choices and select the feasible solution(s) which yield maximum P. This,

---

*The intersection of a finite number of closed half-planes is called a **polygonal convex set. Feasible** region is, therefore, always a **convex polygon.**

however, would usually involve enormous and unnecessary calculations. As an alternative, we use the graphical method to find the maximum(s) of the objective function

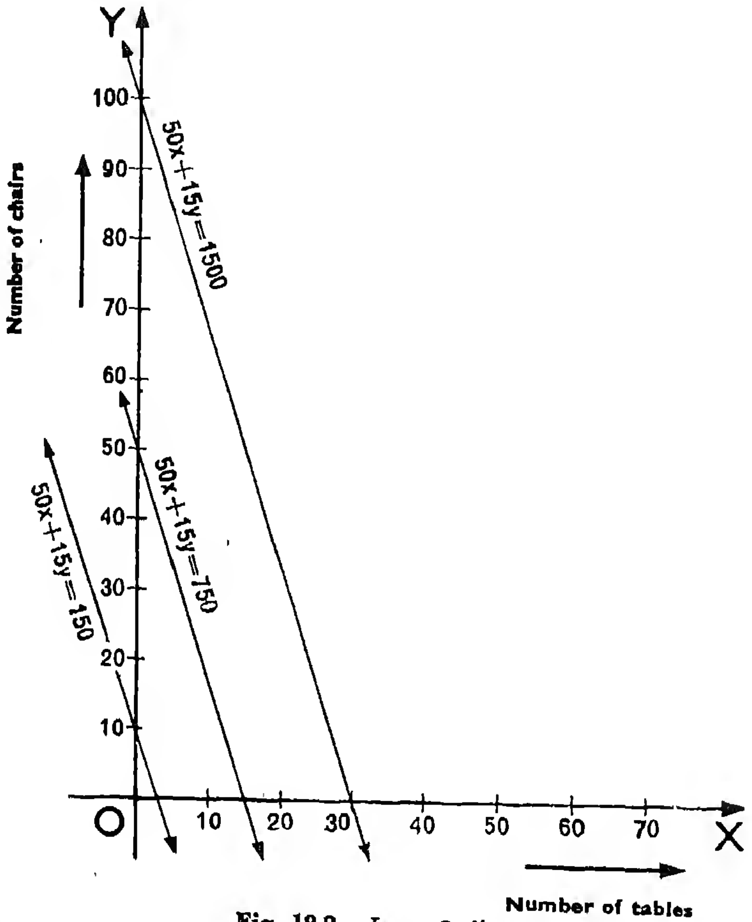


Fig. 18.2. Isoprofit lines

At first sight, the reader may be a bit puzzled as to how to graph the profit function since it is not in the form of an equation. This, however, should cause no concern. All we need to do is to take an arbitrary point, preferably in the feasible region and determine P. For instance, let us take the point (3, 0). P, therefore, is Rs 150.00. (Why ?) It is now easy to graph

$$50x+15y = 150$$

The graph, of course, is a straight line. Further, every point on this line yields a profit of Rs 150.00. No wonder, it is called an **isoprofit line**. (See Fig 18.2). Similarly, we can draw other isoprofit lines with respect to other values of P obtained by taking different points.

The objective function, therefore, represents a set of parallel lines for different values of P. (Why ?) The line $150=50x+15y$ is a member of this set. We observe that the farther

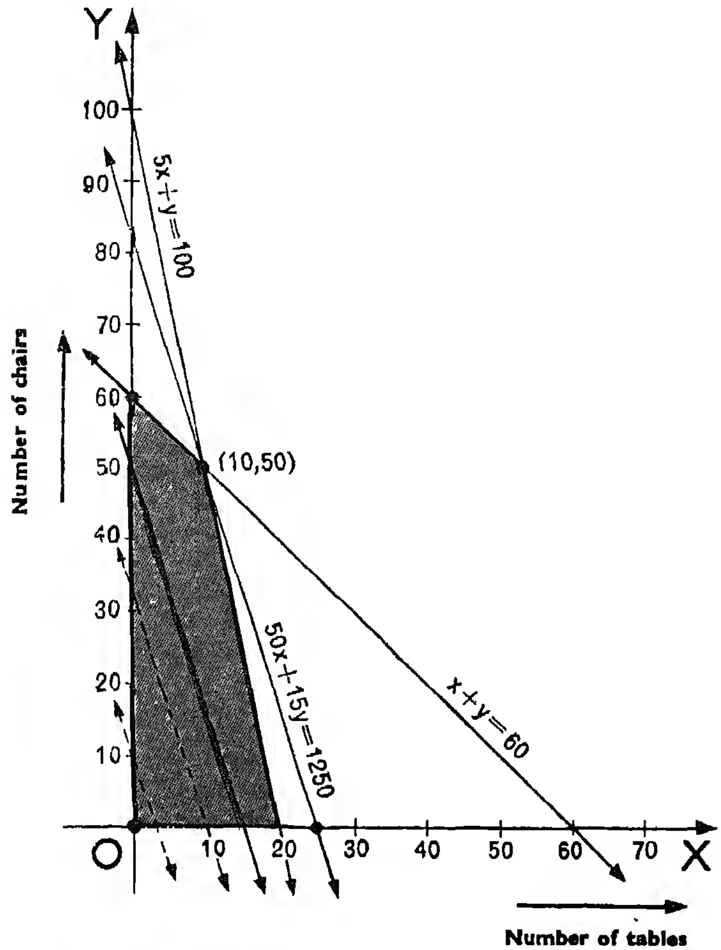


Fig. 18.3. Graphical solution to the Linear Programming Problem of Example 1

the line from the origin, the greater is the value of P. Commonsense dictates, we should continue to draw isoprofit lines to realize higher profits until the line is in a position where it has at least one point in common with the points in the set of feasible solutions and from which position if the line is moved further in the direction away from the origin, the line will not have any point in common with the points in the set of feasible solutions. The feasible solution(s) corresponding to this position of the line gives the maximum* value(s) of the function. (See Fig 18.3)

*__The Fundamental Extreme Point Theorem__ assures us that the maximum (or minimum) of the objective function will occur only at the boundary point(s).

We observe from Fig. 18.3 that the desired position of the isoprofit line is obtained at the vertex* (10,50) of the region.

The optimal investment strategy for the dealer would be, therefore, to invest in 10 tables and 50 chairs. Corresponding to this strategy, his profit would be maximum, namely, Rs 1250.00. (Why ?)

18.4. Some Remarks on the Graphical Method of Solving Linear Programming Problems

Remark 1 : Since the maximum (or minimum) occurs only at the boundary point(s), we need to consider only those positions of the isoprofit line which pass through the corner(s) of the feasible region.

Remark 2 : In case the isoprofit line is parallel to any of the boundary lines of the region, the maximum or (minimum) will occur only at the points of that boundary line.

Remark 3 : Let us calculate the values of P at each of the vertices of the feasible region in Example 1 and represent them in the table below:

Values of P Corresponding to the Vertices of the Feasible Region

Vertex of the Feasible Region	*Corresponding value of P (in rupees)*
(0,0)	0.00
(0,60)	900.00
(10,50)	**1250.00 ← Maximum Profit**
(20,0)	1000.00

We observe that the maximum profit to the dealer results from the investment strategy (10,50).

18.5. Some More Examples on Linear Programming

Example 2 : A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food '1' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food '2' contains 1 unit/kg of vitamin A and 2 unit/kg of vitamin C. It costs Rs 5.00 per kg to purchase food '1' and Rs 7.00 per kg to purchase food '2'. Determine the minimum cost of such a mixture. (Also see Exercise 18, page 79, Part I of this book).

Solution : Let the dietician mix x kg of food '1' and y kg of food '2'. Clearly,

$$x \geqslant 0 \qquad (1)$$

$$y \geqslant 0 \qquad (2)$$

*A **vertex** or **corner** is the point of intersection of two or more boundary lines.

Since the mixture must contain at least 8 units of vitamin A and 10 units of vitamin C, we have the constraints

$$2x+y \geqslant 8 \tag{3}$$

and

$$x+2y \geqslant 10 \tag{4}$$

The objective function or the cost C of the mixture is

$$C = 5x+7y \tag{5}$$

Mathematically, therefore, the problem reduces to minimizing (5) subject to the constraints (1), (2), (3) and (4) As before, we will attempt a graphical solution of this problem. Let us refer to Fig. 18 4

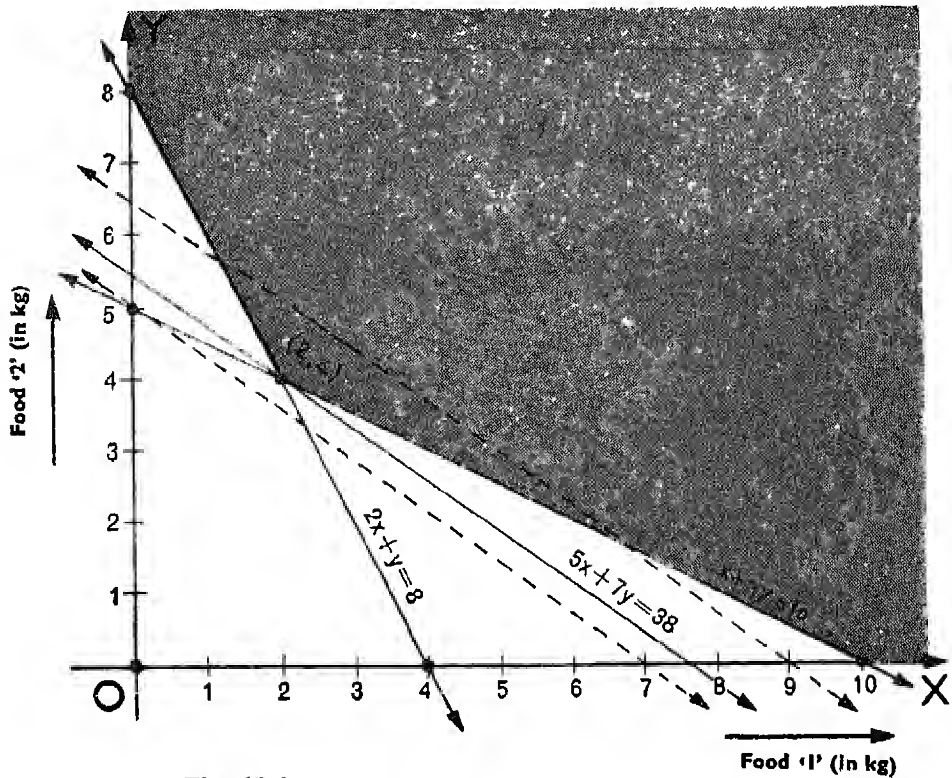


Fig. 18.4. Graphical solution to the Linear Programming Problem of Example 2

The feasible region is the shaded region in the figure. (0, 8), (2, 4) and (10, 0) are its corners. (Why ?)* For different values of C, we will obtain a set of parallel lines. Of course, each line, in this case, would be an **isocost line**. We observe that the nearer the

*(0, 8) and (10, 0) are obvious. (2, 4) is the point of intersection of the two boundary lines $2x+y = 8$ and $x+2y = 10$. The reader is advised to verify for himself by solving the two equations simultaneously.

line to the origin, the smaller is the value of C. We need to concern ourselves with only those positions of the isocost line which pass through the corners of the feasible region. The desired position of the isocost line is obtained at the vertex (2, 4) of the region.

The optimal mixing strategy for the dietician would be, therefore, to mix 2 kg of food '1' and 4 kg of food '2'. Corresponding to this strategy, his cost would be minimum, namely, Rs 38.00. (Why ?)

Example 3 : A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs 2.50 per package on nuts and Re 1.00 per package on bolts. How many packages of each should he produce each day so as to maximize his profit, if he operates his machines for at the most 12 hours a day ? (Also see Exercise 15, page 75, Part I of this book)

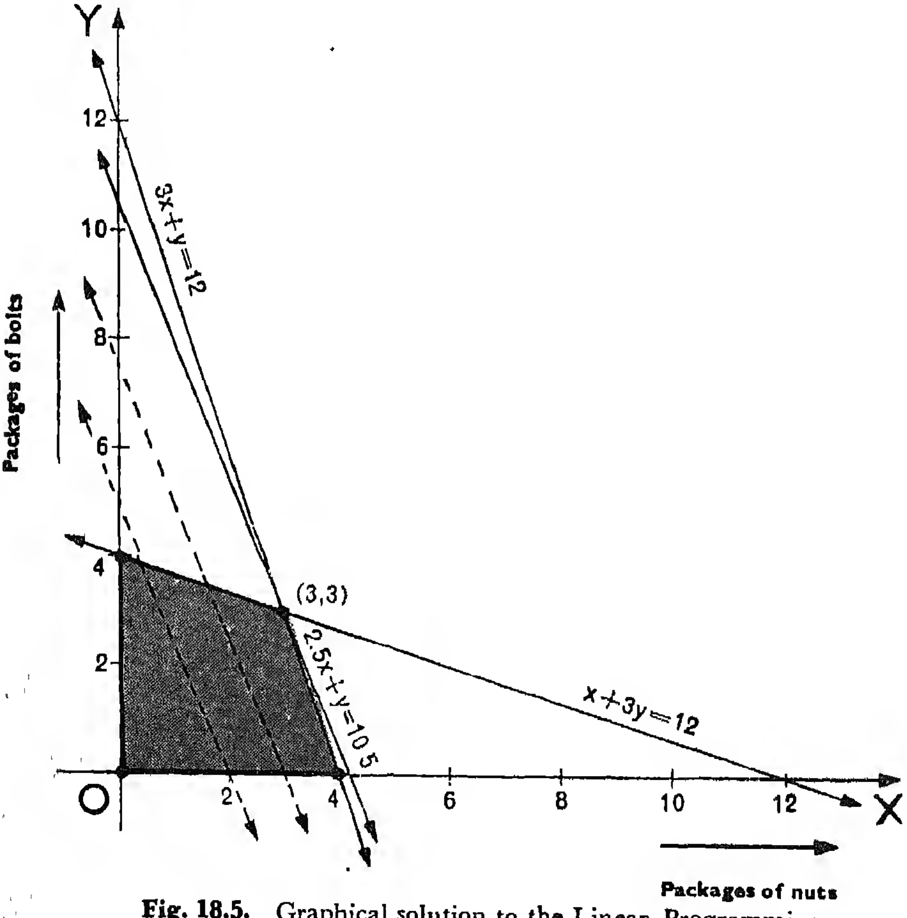


Fig. 18.5. Graphical solution to the Linear Programming Problem of Example 3

Solution : Let the manufacturer produce x packages of nuts and y packages of bolts each day. Clearly,

$$x \geqslant 0 \qquad (1)$$
$$y \geqslant 0 \qquad (2)$$

Since the machines can operate for at most 12 hours a day, we have the constraints corresponding to machines A and B, respectively, as

$$x+3y \leqslant 12 \qquad (3)$$

and
$$3x+y \leqslant 12 \qquad (4)$$

The objective function or the profit, P, is

$$P = 2.5\,x+y \qquad (5)$$

Mathematically, therefore, the problem reduces to maximizing (5) subject to the constraints, (1), (2), (3) and (4) Let us refer to Fig 18.5

The feasible region is the shaded region in the figure. (0, 0), (0, 4), (3, 3) and (4,0) are its corners. For different values of P, we obtain a set of parallel isoprofit lines If we confine ourselves to only those positions of the isoprofit line which pass through the corners of the feasible region, we observe that the isoprofit line through the vertex (3, 3) yields the maximum profit.

The optimal production-strategy for the manufacturer would be, therefore, to manufacture 3 packages each of nuts and bolts daily. Corresponding to this strategy, his profit would be maximum, namely, Rs 10.50. (Why ?)

***Example 4 : (Transportation Problem—A Simplified Version)**

There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are, respectively, 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are, respectively, 8 and 6 units. The cost of transportation per unit is given below

To / *From*	*Cost (in rupees)*		
	A	*B*	*C*
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum ?

Solution : Let x and y units be transported from the factory at P to the depots at A and B, respectively. Then $(8-x-y)$ units will be transported from the factory at P to the depot at C. (Why ?) Clearly,

$$x \geqslant 0 \qquad (1)$$
$$y \geqslant 0 \qquad (2)$$
and
$$8-x-y \geqslant 0 \qquad (3)$$

The weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5-x)$ units need to be transported from the factory at Q. Obviously

$$(5-x) \geqslant 0 \qquad (4)$$

Similarly, $(5-y)$ and $[4-(8-x-y)]$ units need to be transported from the factory at Q. to the depots at B and C respectively. Also

$$(5-y) \geqslant 0 \qquad (5)$$
and
$$[4-(8-x-y)] \geqslant 0$$
i.e.,
$$x+y-4 \geqslant 0 \qquad (6)$$

The objective function or the transportation cost T is

$$T = 16x+10y+15(8-x-y)+10(5-x)+12(5-y)+10(x+y-4)$$
or
$$T = x-7y+190 \qquad (7)$$

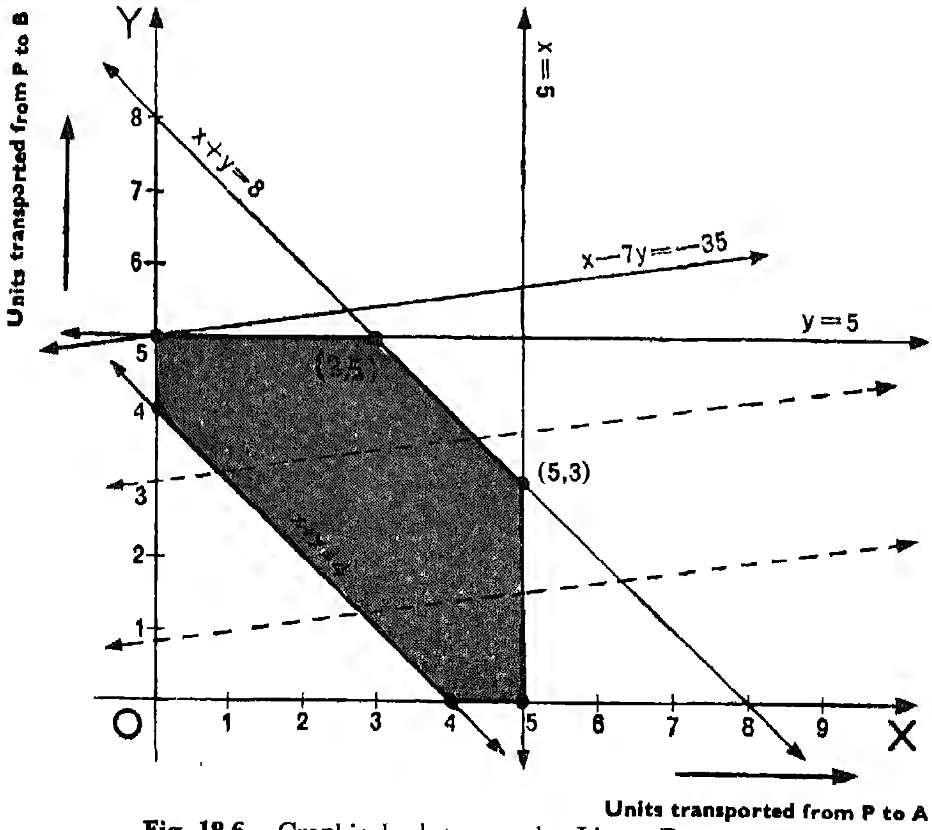


Fig. 18.6. Graphical solution to the Linear Programming Problem of Example 4

Mathematically, therefore, the problem reduces to minimizing (7) subject to the constraints (1), (2), (3), (4), (5) and (6). Let us refer to Fig. 18 6

(0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0) are the corners of the feasible region. For different values of T, we obtain a set of parallel isocost lines. Again, confining ourselves to only those positions of the isocost line which pass through the corners of the feasible region, we observe that the isocost line through the vertex (0, 5) yields the minimum cost.

The optimal transportation strategy would be, therefore, to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C, respectively. Corresponding to this strategy, the transportation cost would be minimum, namely, Rs 155.00. (Why ?)

(The reader is advised to construct a table of values of T corresponding to each vertex of the feasible region and verify, as in Remark 3, Section 18 4.)

Exercise 18.1

1. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs 5760.00 to invest and has space for at most 20 items. A fan costs him Rs 360.00 and a sewing machine Rs 240.00. His expectation is that he can sell a fan at a profit of Rs 22 00 and a sewing machine at a profit of Rs 18.00.

   Assuming he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ?

2. Maximize $P = 3x+4y$ subject to the constraints

$$x \geqslant 0$$
$$y \geqslant 0$$
and
$$x+y \leqslant 4$$

3. Minimize $C = 3x+y$ subject to the constraints

$$x \geqslant 0$$
$$y \geqslant 0$$
$$x+y \leqslant 2$$
and
$$y+4x \leqslant 5$$

*4. A manufacturer has 3 machines installed in his factory Machines I and II are capable of being operated for at most 12 hours whereas machine III must operate at least 5 hours a day. He produces only two items, each requiring the use of the three machines. The number of hours

required for producing 1 unit each of the items on the three machines is given in the following table :

Item	*Number of hours required by the machine*		
	I	*II*	*III*
A	1	2	1
B	2	1	$\frac{5}{4}$

He makes a profit of Rs 6 00 on item *A* and Rs 4.00 on item *B*. Assuming he can sell all that he produces, how many of each item should he produce so as to maximize his profit ? Determine his maximum profit.

5. Two tailors *A* and *B* earn Rs 15.00 and Rs 20.00 per day respectively. *A* can stitch 6 shirts and 4 pants while *B* can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce (at least) 60 shirts and 32 pants at a minimum labour cost ?

6 A factory manufactures 2 types of screws, *A* and *B*, each type requiring the use of two machines—an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machine to manufacture a package of screws '*A*', while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a package of screws '*B*'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a package of screws '*A*' at a profit of 70 p and screws '*B*' at a profit of Re 1.00.
Assuming he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day, in order to maximize his profit ? Determine the maximum profit.

7. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp while it takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at most 20 hours and the grinding/cutting machine for at most 12 hours. The profit from the sale of a lamp is Rs 5.00 and of a shade is Rs 3.00.
Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximize his profit ?

*8. Two godowns A and B have a grain-storage capacity of 100 quintals and 50 quintals, respectively. They supply to 3 ration shops D, E and F whose requirements are 60, 50 and 40 quintals, respectively. The costs of transportation per quintal from the godowns to the shops are given in the following table:

Transportation costs per quintal (in rupees)		
	From	
To	*A*	*B*
D	6.00	4.00
E	3.00	2.00
F	2.50	3.00

How should the supplies be transported in order that the transportation cost is minimum ?

*9. An oil company has two depots, A and B, with capacities of 7000 litres and 4000 litres, respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500, 3000 and 3500 litres, respectively. The distance (in km) between the depots and the petrol pumps is given in the following table:

Distance (in km)		
	From	
To	*A*	*B*
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost/km is 1 paisa per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

18.6. The reader must have realized by now that the mathematical formulation of the linear programming problems invariably leads to a system of linear equations/inequations and polygonal convex sets. The theory of linear inequalities and convex sets has developed

extensively in the past hundred years. However, the field of linear programming problems is fairly recent in its origin

The first problem in linear programming was formulated in 1941 by the Russian mathematician L. Kantorovich and the American economist F.L. Hitchcock, both of whom worked at it independently of each other This is the well-known **transporation problem**. In 1945, an English economist, G. Stigler, described yet another linear programming problem—that of determining an optimal diet.

In the Second World War, when the war operations had to be planned to economize on expenditure, minimize losses and maximize damage to the enemy, linear programming problems came to the forefront. However, they remained top-secret until after the war, when in 1947, an American economist, G.B. Dantzig published a paper in the famous journal *Econometrica* wherein he formulated the general linear programming problem George Dantzig is also credited with using the term 'Linear Programming' and for the solution of the problem by analytical methods.

Linear programming is now widely used in planning economic activity and is an accepted tool in the formulation of national plans.

For his work on these problems, the Russian mathematician, L. Kantorovich, was awarded Nobel Prize for Economics, in the year 1974, together with another famous (American) mathematical-economist, T.C. Koopmans.

CHAPTER XIX

INTRODUCTION TO MATRICES

19.1. Matrix as a Store of Information

Suppose we wish to express the information that Ram has 10 books. We could simply write the number 10 and, perhaps, enclose it within brackets, with the understanding that it is in reference to the number of books Ram possesses. If, from his pocket money, he buys 2 more books, Ram will then have 12 books. Again, with the same understanding, the number 12, perhaps, within brackets, can be used to convey this information In each case, we note that a single number suffices to convey the information.

However, what if Ram has 10 books as also 5 pencils ? How do we convey this information ? We will, clearly, need two numbers. Further, we will have to agree, for instance, that the first number represents the number of books and the second the number of pencils. We could, therefore, write

$$\begin{pmatrix} 10 & 5 \end{pmatrix}$$

Now, if Ram buys 2 more books and 3 more pencils, he will have 12 books and 8 pencils. With the same assumptions, how would we convey this information ? Clearly, we would write

$$\begin{pmatrix} 12 & 8 \end{pmatrix}$$

Let us consider a slightly different situation Ram has 10 books and 5 pencils while his friend Rahim has 6 books and 3 pencils. This information could be displayed in a table as

	Books	*Pencils*
Ram	10	5
Rahim	6	3

We can further abbreviate the display as follows :

$$\begin{array}{cl} \begin{pmatrix} 10 & 5 \\ 6 & 3 \end{pmatrix} & \begin{array}{l} \leftarrow \quad \text{First row} \\ \leftarrow \quad \text{Second row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{First} & \text{Second} \\ \text{column} & \text{column} \end{array} & \end{array}$$

Implied in the above display are the following assumptions .

(*i*) The entries in the first row represent the objects that Ram possesses.

(*ii*) The entries in the second row represent the objects that Rahim possesses.

(*iii*) The entries in the first column represent the number of books.

(*iv*) The entries in the second column represent the number of pencils.

Thus, the entry in the first row and the second column represents the number of pencils that Ram possesses.

(The reader is advised to interpret each entry in the above display.)

We see, therefore, how conveniently the given information can be represented in the form of an arrangement of numbers in rows and columns.

An arrangement of numbers, in rows and columns, is called a MATRIX. Each number is called an ELEMENT or ENTRY of the matrix.

In the matrix

$$\begin{pmatrix} 10 & 5 \\ 6 & 3 \end{pmatrix}$$

the number of rows is 2 and the number of columns is 2. We say that the **order of the above matrix is 2×2 (read as '2 by 2')**. At times, we also say that the above is a **2×2 matrix**

What is the order of the matrix $(10 \quad 5)$? Clearly it has 1 row and 2 columns. The order is, therefore, 1×2. **While specifying the order of a matrix, the number of rows is always written first followed by the number of columns.**

An English mathematician James Joseph Sylvester (1814-1897), who was very fond of assigning imaginative names to his creations and inventions, in 1850, wrote down certain terms and expressions in the form of a rectangular arrangement. Sylvester gave this rectangular arrangement the name **matrix**. Soon, other mathematicians recognized how conveniently a matrix could be employed to extend the notions of number. In particular, the names of Sir William Rowan Hamilton (1805—1865) and Arthur Cayley (1821—1895) deserve special mention. Sir Hamilton, in 1853, and Arthur Cayley, in 1858, made significant contributions to the theory of matrices.

It is fascinating to note that Cayley was called to the bar in 1849. He practiced law for fourteen years. In these fourteen years, Cayley published between 200 and 300 papers in mathematics, many of which are now classics.

Having originated as mere stores of information, matrices have now come to play a very vital role not only in mathematics, but also in biology, sociology, economics, quantum mechanics, psychology and statistics.

Exercise 19.1

1. Mohan, a fruit-seller, is carrying 240 apples and 360 oranges. His competitor Iqbal is carrying 180 apples and 420 oranges. Express this information in the form of a matrix and specify its order.

2. Ganga gets a weekly allowance of Rs 2.00 while her friend Asha Rs 1.50. Express the information as a matrix and specify its order

*3 Consider the following information pertaining to the results of monthly test in class IX in a school .

	Mathematics	*Physics*	*Chemistry*
Pass	31	29	33
Fail	6	8	4

(i) Represent the above information in the form of a 2×3 matrix. Denote the matrix by M and answer the following :

(a) How many rows does M have ? How many columns ?

(b) What do the entries in the first row represent ? Entries in the third column ?

(c) What does the entry in the second row and third column represent? In the first row and second column ?

(ii) Represent the above information in the form of a 3×2 matrix. Denote the matrix by N and answer the following :

(a) What do the entries in the second column represent ? Entries in the first row ?

(b) What does the entry in the first row and second column represent ?

4. Specify the order of each of the following matrices :

(i) $\begin{pmatrix} 3 & 2 & 1 \\ 16 & 0 & 10 \\ -3 & 0 & 8 \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{9}{5} & \frac{8}{5} \\ 0 & -1 \end{pmatrix}$

(iii) $\begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$ (iv) $\begin{pmatrix} 3 \\ 2 \\ 1 \\ -8 \end{pmatrix}$

(v) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (vi) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

19.2. Route Matrices

Let us see some other instances where information can be conveniently stored in the form of a matrix.

Example 1 : There are two rail routes between Delhi and Tundla—one via Aligarh and the other via Agra Fort. And, there is one rail route between Tundla and Kanpur.

Further, there are no routes between Delhi and Kanpur, except through Tundla. Finally, all routes are in both directions . (See Fig. 19 1)

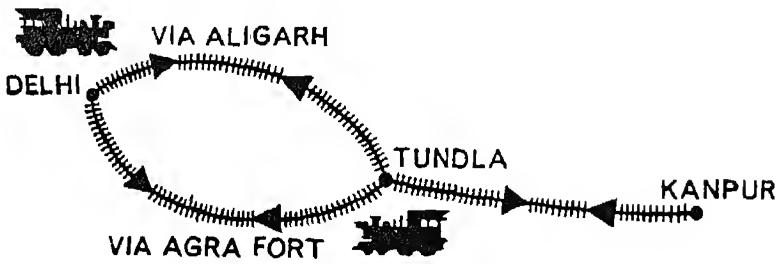


Fig. 19.1

The given information can be represented in a table as follows :

	Number of Routes		
	From		
To	*Delhi*	*Tundla*	*Kanpur (other than through Tundla)*
Delhi	0	2	0
Tundla	2	0	1
Kanpur	0	1	0

In the form of a matrix, the above table can be represented as

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Such a matrix which displays information about the number of routes between places is called a ROUTE MATRIX. The prefix 'Route' simply denotes the fact that the matrix is obtained from data concerning routes between places.

Exercise 19.2

1 Fig 19.2 displays the routes between four places A, B, C and D

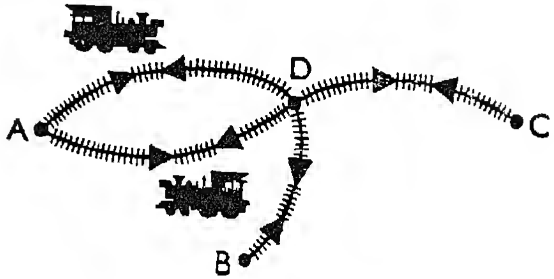


Fig. 19.2

Write the route matrix for the above display

19.3. Incidence Matrices

Let us consider a design consisting of points and line-segments as shown in Fig. 19.3

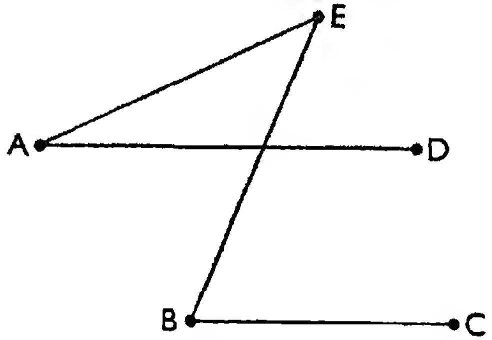


Fig. 19.3

We note there are 4 line-segments AE, AD, EB and BC and 5 points A, B, C, D and E in this design. A 4×5 matrix can, therefore, be used to display the above information, with each of its rows representing a line-segment and each of its columns representing a point. If a point is **incident** on a line (i.e., if a point lies on a line), we write '1' at the intersection of the corrresponding row and column ; otherwise, we write a '0'.

$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} AE \\ AD \\ EB \\ BC \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Since B is incident on the line-segments EB and BC, we see an entry '1' at the intersection of the second column with the third and the fourth row Further, since D is not incident on the line-segments AE, EB and BC, we see an entry '0' at the intersection of the fourth column with the first, third and the fourth row. A matrix such as above is called an **INCIDENCE MATRIX.**

Exercise 19.3

1. Write the incidence matrix for each of the following designs :

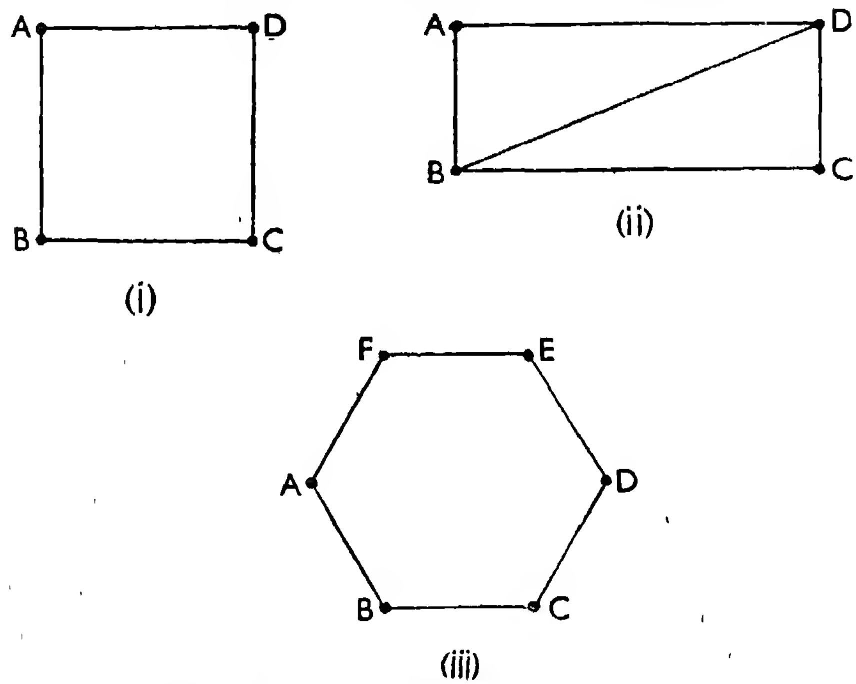


Fig. 19.4

19.4. Operations on Matrices

In earlier classes, we learnt to perform the operations of addition, multiplication, etc. on numbers. We also examined certain properties of these operations, such as commutativity, associativity, etc.

In Part I of this book, we defined certain operations on sets and examined their properties.

We will now learn how we can perform the operations of addition, multiplication, etc. on matrices. But first we need the concept of **equality** of two matrices. **Two matrices A and B are equal if A and B are of the same order (i.e., they have the same number of rows and the same number of columns) and their corresponding elements are equal.** We write, $A = B$.

19.4.1. Addition of Matrices

Example 1 : Sunita has, in her collection, 7 comics and 4 fiction books while her friend Radha has 11 comics and 6 fiction books. We have already seen how to represent this information in the form of a matrix, namely

$$\begin{pmatrix} 7 & 4 \\ 11 & 6 \end{pmatrix}$$

The girls save their weekly pocket allowances and at the end of the month, Sunita purchases 3 comics and 2 fiction books while Radha purchases 1 comic and 1 fiction book. Their purchases can be represented in a matrix as

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

Sunita now has (7+3) or 10 comics and (4+2) or 6 fiction books, while Radha has (11+1) or 12 comics and (6+1) or 7 fiction books. Again, in the form of a matrix, we can write

$$\begin{pmatrix} 10 & 6 \\ 12 & 7 \end{pmatrix}$$

In other words,

$$\begin{pmatrix} 7 & 4 \\ 11 & 6 \end{pmatrix}+\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 7+3 & 4+2 \\ 11+1 & 6+1 \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} 7 & 4 \\ 11 & 6 \end{pmatrix}+\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 10 & 6 \\ 12 & 7 \end{pmatrix}$$

The matrix on the right is the **sum** of the two matrices on the left. Each entry in the matrix on the right is the sum of the corresponding entries in the matrices on the left. Thus, we observe that **the sum of two matrices is a matrix obtained by adding the corresponding elements (or entries) of the given matrices.** We also note that **for the sum to be defined, the matrices have to be of the same order.** It follows that if the matrices are not of the same order, they cannot be added. (Why ?)

Exercise 19.4

1 In an athletic meet between classes IX and X of two schools A and B, the number of students each in high jump and discus throw is given below :

	High jump		*Discus throw*	
	School A	*School B*	*School A*	*School B*
Class IX	4	7	3	5
Class X	6	6	2	8

(a) Write a matrix each, for school A and school B, with rows denoting the 'classes' and columns denoting the 'events'.

(*b*) Can the two matrices be added ? Give reasons for your answer.

(*c*) Find the sum of the two matrices. Interpret each entry in the sum

2 Fill in the blank spaces :

(*a*) $$\begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix}+\begin{pmatrix} 0 & -1 \\ -2 & 5 \end{pmatrix}=\begin{pmatrix} 3 & \dots \\ \dots & \dots \end{pmatrix}$$

(*b*) $$\begin{pmatrix} 5 & 7 & 10 \\ 2 & 0 & 7 \end{pmatrix}+\begin{pmatrix} 1 & 0 & -1 \\ -5 & 1 & -8 \end{pmatrix}=\begin{pmatrix} 6 & \dots & \dots \\ \dots & \dots & -1 \end{pmatrix}$$

(*c*) $$\begin{pmatrix} 2 & \dots \\ \dots & 1 \end{pmatrix}+\begin{pmatrix} \dots & 2 \\ 0 & \dots \end{pmatrix}=\begin{pmatrix} 5 & 4 \\ 1 & 0 \end{pmatrix}$$

3. Perform the indicated operation

(*a*) $$\begin{pmatrix} 8 & 3 \\ -5 & 4 \end{pmatrix}+\begin{pmatrix} 1 & 5 \\ 5 & 0 \end{pmatrix}$$

(*b*) $$\begin{pmatrix} 3 & 2 & 1 \\ -6 & -5 & 3 \\ 0 & 1 & 8 \end{pmatrix}+\begin{pmatrix} -6 & 1 & -2 \\ 3 & 1 & -5 \\ -2 & 1 & 10 \end{pmatrix}$$

(*c*) $$\begin{pmatrix} 6 & -5 & -8 & 6 \\ 3 & 2 & 17 & 4 \\ 6 & 3 & 1 & 3 \end{pmatrix}+\begin{pmatrix} 8 & -2 & 8 & 7 \\ 9 & 3 & 3 & -10 \\ 5 & 6 & 1 & 6 \end{pmatrix}$$

(*d*) $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4. Calculate the sum of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The matrix with each entry '0' is called a **zero matrix** and is, usually, denoted by $\mathbf{0}$. A matrix with at least one non-zero entry is, therefore, called a **non-zero matrix.**

Let us denote the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by A. Then, we note, if $\mathbf{0}$ and A are of the same order,

$$\mathbf{0}+A = A$$

How about $A+\mathbf{0}$? (The reader is advised to verify that $A+\mathbf{0} = A$)

5. If $$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}+\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \quad \text{find} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

6 Let $A = \begin{pmatrix} 3 & -6 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -8 & 9 \end{pmatrix}$

(a) Is $A+B$ defined ? Find $A+B$.

(b) Is $B+A$ defined ? Find $B+A$.

We will note that $A+B = B+A$

In fact, it can be shown that matrix addition is commutative.

7. Let

$$A = \begin{pmatrix} 6 & 1 \\ 3 & 1/2 \\ -5 & 1 \end{pmatrix}.$$ Calculate $A+A$.

It is natural to denote $A+A$ by $2A$ We note that in the matrix $2A$, each entry is two times the corresponding entry in the matrix A. (The reader is advised to calculate $2A+A$, denoted by $3A$. Every entry in $3A$ will be three times the corresponding entry in the matrix A.)

Thus, if α is a real number and A is any matrix, αA is a matrix each of whose entries is α times the corresponding entry in the matrix A. For instance, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

If $\alpha = -1$, we obtain the matrix $(-1)\,A$ = '$-A$', namely

$$\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

It is easy to verify that

$$A+(-A) = \mathbf{0} = (-A)+A$$

8 Fill in the blank places

(a)
$$5\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

(b)
$$6\begin{pmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 12 & \dots & -6 \\ \dots & 12 & \dots \end{pmatrix}$$

(c)
$$\frac{1}{3}\begin{pmatrix} 3 & 6 & -3 \\ 2 & 4 & 12 \\ 0 & -1 & 9 \end{pmatrix} = \begin{pmatrix} \dots & 2 & -1 \\ 2/3 & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

(d)
$$-\frac{2}{3}\begin{pmatrix} 8 & \dots & \dots \\ 3 & \dots & \dots \\ \dots & -12 & \dots \end{pmatrix} = \begin{pmatrix} -16/3 & 1 & 0 \\ -2 & 3 & -4 \\ 6 & 8 & 8 \end{pmatrix}$$

(e)
$$-4\begin{pmatrix} \dots & 0 \\ 0 & \dots \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ \dots & -4 \end{pmatrix}$$

9. Determine p and q such that

$$p\begin{pmatrix} -3 \\ 4 \end{pmatrix} + q\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ -11 \end{pmatrix}$$

10. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 2 \\ 6 & -4 \end{pmatrix}$

(*a*) Find $A+(B+C)$

(*b*) Find $(A+B)+C$

We will note that

$$A+(B+C) = (A+B)+C$$

In fact, it can be shown that **matrix addition is associative.**

19.4.2. Multiplication of Matrices

Example 2 : Joseph and Ali are two friends. Joseph needs to buy 3 pencils and 2 note-books, while Ali needs 6 pencils and 5 note-books. They go to a stationery shop and are quoted the followgin rates :

	Cost (in paise)
Pencil	30
Note-book	40

How much money does each need to spend ? Clearly, Joseph needs $(3\times30+2\times40)$ or Rs 1.70 and Ali needs $(6\times30+5\times40)$ or Rs 3.80. In terms of matrices, we note their

Requirements	*Prices*	*Money needed*
$\begin{pmatrix} 3 & 2 \\ 6 & 5 \end{pmatrix}$	$\begin{pmatrix} 30 \\ 40 \end{pmatrix}$	$\begin{pmatrix} (3\times30+2\times40) \text{ or } 170 \\ (6\times30+5\times40) \text{ or } 380 \end{pmatrix}$

What if there is another stationery shop in their locality which quotes the following rates :

	Cost (in paise)
Pencil	35
Note-book	45

If they want to make their purchases from this shop, how much money would each need to spend ? The reader can easily verify that Joseph would need $(3\times35+2\times45)$ or Rs 1.95 and Ali would need $(6\times35+5\times45)$ or Rs 4.35.

In terms of matrices, we represent the two informations as :

Requirements	*Prices at the two shops*	*Money needed at the two shops*
$\begin{pmatrix} 3 & 2 \\ 6 & 5 \end{pmatrix}$	$\begin{pmatrix} 30 & 35 \\ 40 & 45 \end{pmatrix}$	$\begin{pmatrix} 170 & 195 \\ 380 & 435 \end{pmatrix}$

The above example illustrates **multiplication of matrices.** How are the elements in the product-matrix obtained ? Let us denote by A the requirement-matrix, B the price-matrix and C the money-needed-matrix

To obtain the entry at the intersection of first row and first column of C, the product-matrix,

(i) consider the first row $(3 \quad 2)$ of A, and the first column $\begin{pmatrix} 30 \\ 40 \end{pmatrix}$ of B.

(ii) multiply each entry in the row by the corresponding entry in the column and obtain the sum, namely, $(3 \times 30 + 2 \times 40)$ or 170. 170 is the desired entry.

How about the entry at the intersection of the first row and the second column of C? We would take the first row $(3 \quad 2)$ of A, the second column $\begin{pmatrix} 35 \\ 45 \end{pmatrix}$ of B, multiply the corresponding entries and add to obtain $(3 \times 35 + 2 \times 45)$ or 195.

(The reader is advised to verify for himself that the other two entries in the product-matrix can be obtained similarly)

Since the row entries are multiplied with the corresponding column entries and added to obtain an entry in the product-matrix, **it is necessary that the number of columns in A should be the same as the number of rows in B in order that the product of A and B, written as AB, is defined.** In the product AB, we say that B is **pre-multiplied** by A and that A is **post-multiplied** by B.

Example 3 : Calculate

$$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Solution :

$$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 6 \times 3 & 1 \times 2 + 6 \times 4 \\ 0 \times 1 + 1 \times 3 & 0 \times 2 + 1 \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 26 \\ 3 & 4 \end{pmatrix}$$

Example 4 : Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad \text{Find } AB$$

Solution :

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

We note that **it is possible for two non-zero matrices to have a zero-matrix as their product.**

(The reader is advised to recall that this is not so for real numbers. If a, b are real numbers then $ab = 0$ implies either $a = 0$ or $b = 0$ or both.)

Example 5 : Let

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } F = \begin{pmatrix} 3 & 1 \\ 4 & 9 \end{pmatrix}. \quad \text{Find } DF \text{ and } FD.$$

Solution :

$$DF = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 8 & 18 \end{pmatrix}$$

$$FD = \begin{pmatrix} 3 & 1 \\ 4 & 9 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 8 & 18 \end{pmatrix}$$

We note that $DF = FD$

Example 6 : Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{Find } AC \text{ and } CA$$

Solution :

$$AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$CA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We note that $AC \neq CA$ In fact, it can be shown that **the matrix multiplication is not necessarily commutative.**

Exercise 19.5

1 Fill in the blank spaces

(a) $$\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ -8 & 3 \end{pmatrix} = \begin{pmatrix} -22 & \dots \\ \dots & 2 \end{pmatrix}$$

(b) $$\begin{pmatrix} 3 & 6 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \dots & \dots \\ 2 & \dots \end{pmatrix}$$

2. Calculate

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3. Calculate

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

4 Calculate

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

5 Calculate

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

6 Let $A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$. Find AA or A^2.

7. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}$

(a) Find $A(BC)$

(b) Find $(AB)\,C$

We will note that

$$A\,(BC) = (AB)\,C$$

In fact, it can be shown that **the matrix multiplication is associative**, assuming, of course, the multiplication is defined. This is, indeed, a beautiful property of matrix multiplication. **Even though the matrix multiplication is not necessarily commutative, it is necessarily associative.**

8. Calculate

(a) $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

(b) $$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

9. Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $B = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Determine AB.

*10. Let $A = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$.

Verify that

$$A\,(B+C) = AB+AC$$

It can be shown that **matrix multiplication is distributive with respect to addition**, assuming, of course, the multiplication is defined. In fact, for matrices, we have two distributive laws, namely,

Left Distributivity : $A(B+C) = AB+AC$
Right Distributivity : $(B+C)A = BA+CA$

*19.5. Applications of Matrices

In Part I, Chapter V of this book, we have learnt how to solve simultaneous linear equations by graphing and by elimination We will now use matrices to solve a system of two simultaneous linear equations in two unknowns. We will assume that the system admits a unique solution

Let the two equations in two unknowns be written as

$$\left.\begin{aligned} ax+by &= p \\ cx+dy &= q \end{aligned}\right\} \tag{1}$$

Written in terms of matrices, these are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{2}$$

For obvious reasons $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the **co-efficient matrix and** $\begin{pmatrix} p \\ q \end{pmatrix}$, the **matrix of constants.**

Let us pre-multiply both sides of (2) by the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ We obtain

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

i.e.,
$$\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

or,
$$(ad-bc)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{3}$$

If $(ad-bc) \neq 0$, (only then the system will admit a unique solution), we can divide both sides of (3) by $(ad-bc)$ and obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

How did we obtain the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ with which we pre-multiplied both sides of (2) ? Without proof, we state that the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is obtained from the coefficient-matrix as follows :

(*i*) The positions of '*a*' and '*d*' are interchanged.

(*ii*) The sign of each of the remaining two entries is changed.

Example 1 : Solve
$$2x+3y = 5$$
$$3x+5y = 8$$

Solution : The coefficient matrix is, obviously,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

Thus, $$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$

Furthermore, $(ad-bc) = 10-9 = 1$, which is, of course, not equal to zero.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e., $x = 1, \quad y = 1$

(The reader is advised to verify that $x = 1, \quad y = 1$ is indeed the solution of the above system.)

Example 2 : Solve $$4x+5y = 7$$ $$2x-3y = 8$$

Solution : Here

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 2 & -3 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ -2 & 4 \end{pmatrix}$$

And, $(ad-bc) = -12-10 = -22$

Thus,

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{22}\begin{pmatrix} -3 & -5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{61}{22} \\ -\frac{18}{22} \end{pmatrix}$$

i.e., $x = \frac{61}{22}$ and $y = -\frac{9}{11}$

Exercise 19.6

1. Consider the system of equations
$$2x+5y = 11$$
$$4x-3y = 9$$
   (a) Write the coefficient matrix and the matrix of constants.
   (b) Use matrices to solve the above system of equations.
2. Solve, using matrices, the simultaneous equations
$$3x+4y = 8$$
$$x-6y = 10$$

*19.6. Secret Messages with Matrices

We will now describe a method of sending secret messages by using matrices. You can try out this method with a friend and exchange secret messages with him or her.

Let us assume that each secret message will consist of exactly four letters from the English alphabet. Say, the message you want to send to your friend is ACID. Here is how you could send this message

First replace each letter by its position in the alphabet. For instance, A has position 1, B has 2, C has 3, etc. The message can now be written as 1394.

Next, it can be arranged in the form of a 2×2 matrix as $\begin{pmatrix} 1 & 3 \\ 9 & 4 \end{pmatrix}$. Let us denote this matrix by M.

So, all you need to do is to send the matrix M to your friend, which he can easily 'decode'.

However, this method will soon be figured out by any of your 'spies' who has some imagination. What you need to do, therefore, is to agree with your friend, **in advance**, upon some fixed matrix, for instance,

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

This matrix is to be used for all messages between you and your friend and should remain a secret between you two. A note of caution, however. If you view the matrix A as the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(ad-bc)$ must work out to be non-zero. In our case, we observe that, $ad-bc = 3(1)-(-1)(2) = 5$

Now send to your friend the matrix AM rather than M as the secret message. In other words, send him the matrix

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 21 & 17 \\ 8 & 1 \end{pmatrix} = M_1 \text{ (say)}$$

He, of course, knows the secret matrix A agreed upon between you two in advance. He also knows that you will send him $AM = M_1$ rather than M. How does he 'decode' it? He will pre-multiply M_1 by the matrix $\frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ and obtain

$$\frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 21 & 17 \\ 8 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 15 \\ 45 & 23 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 9 & 4 \end{pmatrix}$$

The message is, therefore, 1394 or ACID.

MISCELLANEOUS EXERCISE
(On Chapter XIX)

1. The number of primary, middle and higher secondary schools in India, in 1966, was 391064, 75798 and 27477 respectively ; in 1972, was 414406, 94199 and 38488 respectively. Express the above information in the form of a matrix.
2. In 1972-73 (fiscal year), the expenditure of the Department of Atomic Energy was Rs 25.16 crores, of the Department of Space was Rs 18 23 crores and of the Defence Research was Rs 25.35 crores. In 1973-74 (fiscal year), their expenditures were respectively Rs 22.63 crores, Rs 19.09 crores and Rs 31.51 crores. Express this information in the form of a matrix.
3. There are four cities P, Q, R and S such that there are 3 direct links between P and R, 2 between P and Q, none between Q and R, one each between P and S, between Q and S and between R and S. Write the consequent route matrix
4. Write the incidence matrix for the following design :

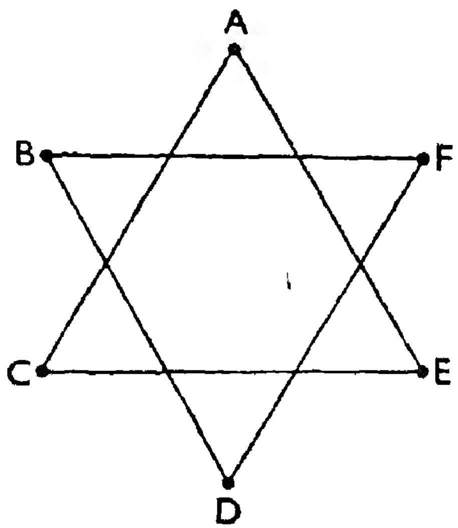


Fig. 19·5

5. Perform the indicated operation(s) :

(a) $\begin{pmatrix} 3 & 2 \\ 9 & -1 \\ 8 & 2 \end{pmatrix} + \begin{pmatrix} -8 & 17 \\ 8 & 17 \\ -6 & -18 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 5 & 4 & -8 & 2 \\ 3 & 2 & 1 & 7 & 1 \end{pmatrix} + \begin{pmatrix} -7 & 6 & 5 & -6 & -1 \\ 3 & -1 & 2 & 0 & 0 \end{pmatrix}$

(c) $-\frac{1}{7}\begin{pmatrix} 21 & 35 \\ 70 & 140 \\ 2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 6 & -1 \\ 8 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 9 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 6 & 1/8 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

6. Find a value of x such that

$$\begin{pmatrix} x & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

7. Write the following system of equations in matrix form

(a) $3x+8y = 7$
$6x- y = 31$

(b) $18x+25y = 7$
$2x-11y = 9$

8. Solve each of the following system of equations with the use of matrices

(a) $2x+5y = 19$
$x+ y = 5$

(b) $7x-9y = -29$
$x+8y = 33$

*9. Given the matrices A and B, find AB and BA.

$$A = \begin{pmatrix} -5 & 0 & 3 \\ 2 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 1 & 0 \\ -2 & 3 & 5 \end{pmatrix}$$

CHAPTER XX

QUADRATIC POLYNOMIALS, EQUATIONS AND INEQUATIONS

20.1. Polynomials

We are already familiar with expressions, such as, $8x$, $y^2+9xy+3y$, $3x^3+7x^2$, z^2-x^2, k^3-8k^2+1, called **polynomials** $8x$, $3x^3+7x^2$, k^3-8k^2+1 are examples of **polynomials in one variable**. $y^2+9xy+3y$, z^2-x^2 are examples of **polynomials in two variables.**

In this chapter, we shall limit our discussion to polynomials in one variable only.

The degree of a polynomial, in one variable, is the greatest exponent or power (whole number) of the variable. For instance, $8x$ is a polynomial of degree 1, $\sqrt{2}x^2+3x$ is of degree 2, $3x^3+7x^2$ and k^3-8k^2+1 are each of degree 3, etc. 5 is a polynomial of degree 0.(Why ?)

The simplest of the polynomials are those which contain only one term, for instance, $8x$, 16, $\sqrt{3}y^2$, etc. They are called **monomials** A polynomial containing two terms is called a **binomial** while the one containing three terms is called a **trinomial.** $x^2+\frac{1}{7}x$, $y+9$, $\sqrt{2}\,z^2+\frac{1}{3}$, etc. are examples of binomials. x^2+3x+2, $y^2+9y+\frac{1}{2}$, $z^5+3z+\frac{\sqrt{3}}{2}$, etc are examples of trinomials.

We will further limit our discussion to polynomials of degree two only.

A polynomial (in one variable) of degree one is called a **linear polynomial**. A polynomial of degree two, in one variable, is called a **quadratic polynomial**. We note that a quadratic polynomial, in one variable, consists of at most three terms. The general form of a quadratic polynomial (over R) in one variable, say x, is

$$ax^2+bx+c$$

where a, b and $c \in R$, and $a \neq 0$. a is the **coefficient** of x^2, b that of x and c is the **constant term.**

(We assume that the reader is already familiar with the addition, subtraction, multiplication and division of polynomials.)

20.2. Polynomial Functions

Let us consider the polynomial x^2-2x+3. We note that for each real x, the polynomial x^2-2x+3 admits a unique real value For instance,

if $x = 0$, $x^2-2x+3 = 3$,

if $x = -1$, $x^2-2x+3 = 6$,

if $x = 3/4$, $x^2-2x+3 = 33/16$, etc.

Thus the polynomial x^2-2x+3 defines a function whose domain is R and whose range is also R. The polynomial function corresponding to the given polynomial x^2-2x+3 is written as

$$f(x) = x^2-2x+3$$

The value of a polynomial, in x, for a given (real) x is the value of the corresponding polynomial function at the given (real) x The **graph of a polynomial** is the graph of the corresponding polynomial function.

20.3. Graphical Representation of Quadratic Polynomials

Example 1 : Draw the graph of the quadratic polynomial $4x^2$

Solution : The polynomial function corresponding to the polynomial $4x^2$ is, of course,

$$f(x) = 4x^2$$

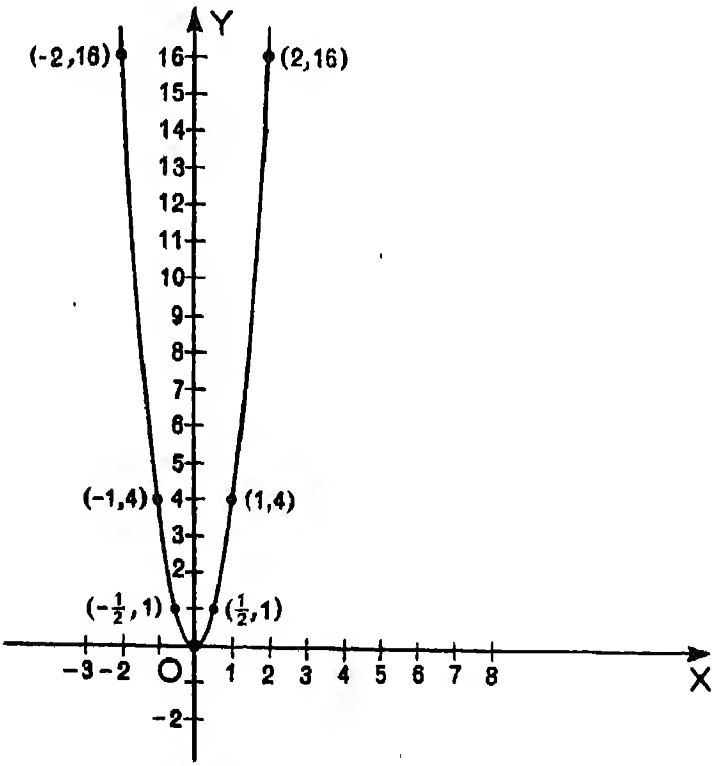


Fig. 20.1. Graph of $f(x) = 4x^2$

For each x, we obtain a unique $f(x)$. We, therefore, have ordered pairs $(x, f(x))$ which we can easily plot in the cartesian plane. We usually plot* x along the x-axis and $f(x)$ along the y-axis. Often, enough ordered pairs are plotted until the pattern of the plot is clear and the resulting points are then joined by a free-hand curve.

We prepare a table of selected values of x and $f(x)$.

x	0	-1	-2	1	2	1/2	$-1/2$
$f(x)$ or y	0	4	16	4	16	1	1

The graph of the polynomial $4x^2$ is given in Fig. 20.1. We note that the curve meets the x-axis at one point only, namely, $(0, 0)$.

Example 2 : Draw the graph of the polynomial $9-x^2$

Solution : The polynomial function corresponding to the polynomial $9-x^2$ is

$$f(x) = 9-x^2$$

As before, we prepare a table of selected values of x and $f(x)$

x	$f(x)$ or y
3	0
-3	0
2	5
-2	5
4	-7
-4	-7
0	9
1	8
-1	8

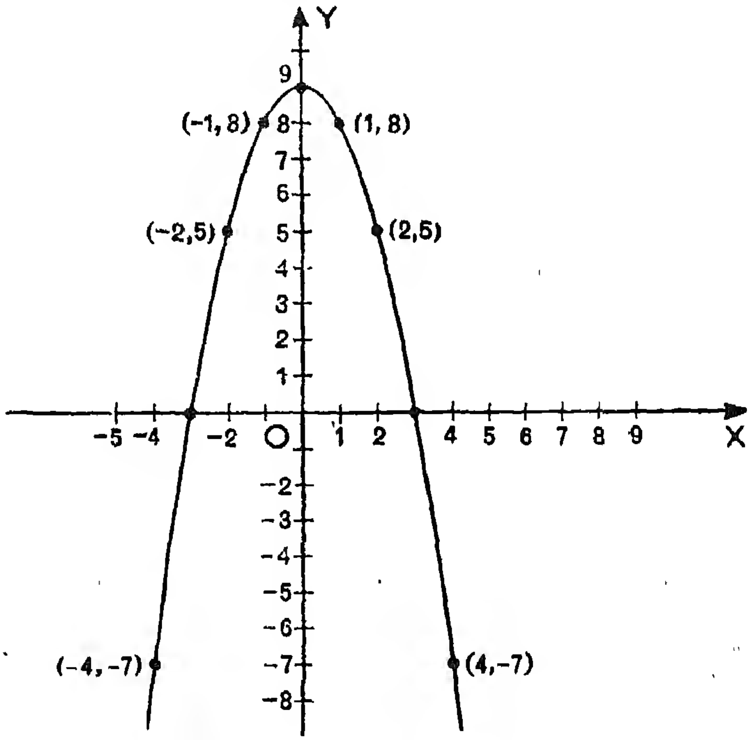


Fig. 20.2. Graph of $f(x) = 9-x^2$

*For the sake of convenience, we usually write $y = 4x^2$ rather than $f(x) = 4x^2$.

The graph of the polynomial $9-x^2$ is given in Fig. 20.2 We note that the curve intersects the x-axis in two distinct points, namely, (3, 0) and (—3, 0).

Example 3 : Draw the graph of the quadratic polynomial $2x^2-5x+2$

Solution : The polynomial function corresponding to the polynomial $2x^2-5x+2$ is

$$f(x) = 2x^2-5x+2$$

We prepare a table of selected values of x and $f(x)$.

x	$f(x)$ or y
-1	9
0	2
$-\frac{1}{2}$	5
$\frac{1}{2}$	0
1	-1
2	0
3	5
$\frac{7}{2}$	9

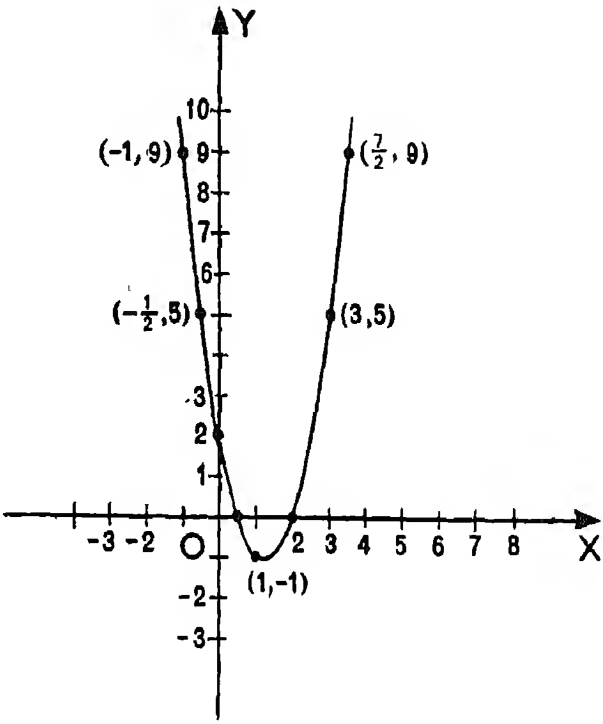


Fig. 20.3. Graph of $f(x) = 2x^2-5x+2$

The graph of the polynomial $2x^2-5x+2$ is given in Fig. 20 3 We note that the curve intersects the x-axis in two distinct points, namely, ($\frac{1}{2}$, 0) and (2, 0).

Example 4 : Draw the graph of the polynomial x^2-2x+3

Solution : The polynomial function corresponding to the polynomial x^2-2x+3 is

$$f(x) = x^2-2x+3$$

We prepare a table of selected values of x and $f(x)$

The graph of the polynomial x^2-2x+3 is given in Fig. 20.4 We note that the curve

x	$f(x)$ or y
-2	11
-1	6
0	3
1	2
2	3
3	6
4	11

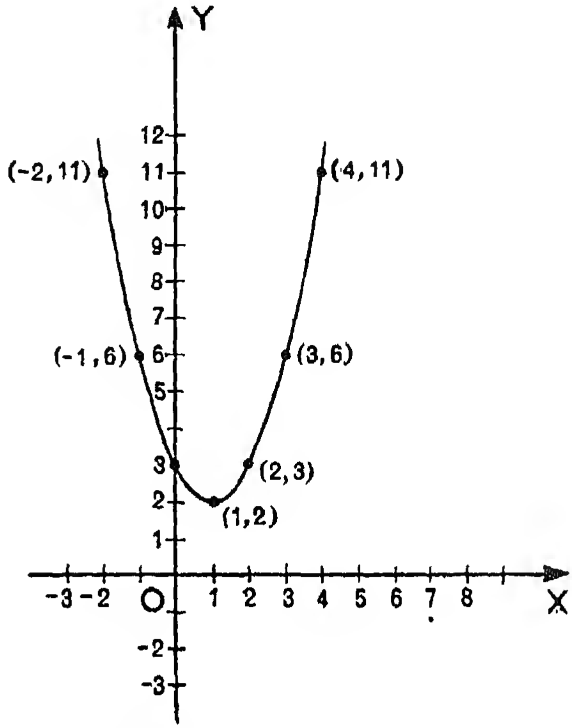


Fig. 20.4. Graph of $f(x) = x^2-2x+3$

does not meet or intersect the x-axis at all.

20.4. Zeros of a Polynomial

Let us refer, say, to the polynomial $9-x^2$. (See Example 2, Section 20.3.) For what value(s) of x does this polynomial have a value zero ? Clearly, for $x = 3$ and $x = -3$. 3 and -3 are said to be the **zeros** of the polynomial $9-x^2$.

A real number r is said to be a zero of a polynomial, in x, if the value of the polynomial at x=r is zero, i.e., if the polynomial vanishes at x = r. How do we determine graphically the zero(s) of a polynomial ?

We know that the y-coordinate (ordinate) of a point on the x-axis is zero. Therefore, if the graph of a polynomial function intersects (or meets) the x-axis at the point (r, 0), the polynomial vanishes at $x = r$. Conversely, if the polynomial vanishes at $x = r$, the graph of the corresponding polynomial function intersects (or meets) the x-axis at the point $(r, 0)$. Hence, **the x-coordinates (abscissae) of the points of intersection (or contact) of the graph of a polynomial function and the x-axis are precisely the zeros of the polynomial.**

(The reader is advised to note that the determination of the zeros from a graph will, at times, yield only approximate values)

We give below the zero (s) of the polynomials of Examples 1 to 4 ·

Polynomial	*Zeros*	*Sign of the coefficient of x^2 in the polynomial*	*Parabola opens*
$4x^2$	0	Positive	Upwards
$9-x^2$	$-3, 3$	Negative	Downwards
$2x^2-5x+2$	$\frac{1}{2}, 2$	Positive	Upwards
x^2-2x+3	None	Positive	Upwards

Let us examine their graphs and make some observations .

(1) The graph of each of the polynomials is essentially of the same shape. The graph is known as a **parabola.** In fact, it can be shown that **the graph of a quadratic polynomial over R is a parabola.**

(2) The parabola opens upwards if the sign of the coefficient of x^2 is positive and opens downwards if the sign of the coefficient of x^2 is negative

Exercise 20.1

Draw the graph of each of the following polynomials. Read the zero(s) from the graph.

1. x^2-36
2. $4-\frac{1}{2}x^2$
3. $-2x^2-3x+7$
4. $10x^2+\frac{4}{3}x$
5. $3x^2+\frac{9}{4}x+12$
6. $x^2-8x+16$

20.5. Factors of a Quadratic Polynomial

Let us take two linear polynomials

$$3x+1 \quad \text{and} \quad 2x+3$$

We know how to multiply them We also know that their product is of degree 2 and hence, a quadratic polynomial.

$$(3x+1)(2x+3) = 6x^2+11x+3$$

We say that $3x+1$ and $2x+3$ are **factors** of the quadratic polynomial $6x^2+11x+3$.

We prove that **the product of any two linear polynomials is a quadratic polynomial.**

Let us denote the linear polynomials as $lx+m$ and $px+q$. Of course, $l \neq 0$ and $p \neq 0$.(Why ?). Then

$$\begin{aligned}(lx+m)\,(px+q) &= lpx^2+(lq+mp)x+mq \\ &= ax^2+bx+c \quad \text{(say)}\end{aligned}$$

Again $lp \neq 0$.(Why ?) The degree of the product is 2 and hence, the product is a quadratic polynomial.

$lx+m$ and $px+q$ are said to be the **factors** of the quadratic polynomial.

Now, let us look at the converse problem of finding two linear polynomials, if possible, whose product is the given quadratic polynomial. This process is called **factoring** or **factorizing** the quadratic polynomial. Once, we find the two linear polynomials, we say the given quadratic polynomial has been factorized into linear factors.

Example 1 : Factorize $x^2-7x+12$

Solution : We wish to express $x^2-7x+12$ as a product of two linear polynomials, say, $lx+m$, $(l \neq 0)$ and $px+q$, $(p \neq 0)$.

i.e.,
$$\begin{aligned}x^2-7x+12 &= (lx+m)(px+q) \\ &= lpx^2+(lq+mp)x+mq\end{aligned}$$

Thus, $$lp = 1,\ lq+mp = -7 \text{ and } mq = 12$$

We wish to find four numbers l, m, p and q such that the above three conditions are satisfied This is usually quite tedious What we, therefore, do is the following :

We note that the product of the coefficient of x^2 and the constant term is $lpmq$ or $(lq)(mp)$. The coefficient of x is $(lq+mp)$. **We, therefore, look for two numbers, say, r(=lq) and s(=mp) such that**

$$\mathbf{r+s = \text{coefficient of } x}$$

and $$\mathbf{rs = (\text{coefficient of } x^2)\times(\text{constant term})}$$

This, of course, will be much less tedious. For the example above, we need r and s such that

$$\begin{aligned}r+s &= -7 \quad \text{and} \\ rs &= 12\end{aligned}$$

A moment's reflection will show that the two numbers are -3 and -4. It is, of course, immaterial which number is denoted by r and which by s. We, therefore, split the term containing x into two terms and write

$$\begin{aligned}x^2-7x+12 &= x^2-3x-4x+12 = x(x-3)-4(x-3) \\ &= (x-3)(x-4)\end{aligned}$$

Hence, the polynomial $x^2-7x+12$ has been factorized into linear factors $(x-3)$ and $(x-4)$.

Example 2 : Factorize $5x^2+13x+6$

Solution : Here the product of the coefficient of x^2 and the constant term is 5×6 or 30. We, usually, indicate this as follows :

$$5x^2+13x+6$$

$$30$$

Thus, we seek two numbers r and s, such that

$$r+s = 13 \quad \text{and}$$
$$rs = 30$$

A little trial and error will give us the two numbers as 10 and 3. We write

$$5x^2+13x+6 = 5x^2+10x+3x+6 = 5x(x+2)+3(x+2)$$
$$= (x+2)(5x+3)$$

Thus, $$5x^2+13x+6 = (x+2)(5x+3)$$

Is it always possible to factorize a quadratic polynomial over R into two linear polynomials over R ? Let us see

Example 3 : Factorize x^2+x+1 (over R)

Solution : We want r and s, such that

$$r+s = 1 \quad \text{and}$$
$$rs = 1$$

We can easily see that there are no two real numbers whose sum and product is each 1. Thus, x^2+x+1 cannot be factorized over R

20.6. Conditions for Factorization of a Quadratic Polynomial (over R) into Linear Factors

We have seen from Example 3 above that it is not necessary for every quadratic polynomial (over R) to admit factorization into linear factors (over R). Let us see when can this happen.

ax^2+bx+c can be factorized into two linear factors, if and only if, we can find two real numbers r and s such that

$$r+s = b \quad \text{and}$$
$$rs = ac$$

Now, $$(r-s)^2 = (r+s)^2-4rs \qquad \text{(Why?)}$$
$$= b^2-4ac$$

Since, $(r-s)^2 \geqslant 0$ (Why ?), $b^2-4ac \geqslant 0$

Thus, a quadratic polynomial over R can be factorized into linear factors over R, if and only if, b^2-4ac is non-negative.

In Examples 1 and 2 of Section 20.5, this was indeed the case. $b^2-4ac = 49-48 = 1$ (Why ?) in Example 1 and $b^2-4ac = 169-120 = 49$ in Example 2.

In Example 3, however

$$b^2-4ac = -3 < 0$$

20.7. Formula for Factorizing a Quadratic Polynomial into Linear Factors

Consider the polynomial ax^2+bx+c, $a, b, c \in R$ and $a \neq 0$. As remarked in Section 20.6, ax^2+bx+c can be factorized into linear factors over R, if and only if, (1) $b^2-4ac>0$ or (2) $b^2-4ac = 0$. Let us examine these possibilities one by one

Case 1 : $b^2-4ac > 0$

Two real numbers r and s such that $r+s = b$ and $rs = ac$ are

$$\frac{b+\sqrt{b^2-4ac}}{2} \text{ and } \frac{b-\sqrt{b^2-4ac}}{2}$$

(The reader is referred to Appendix VI A for the derivation of r and s)

The quadratic polynomial can be written as a product of two linear factors as

$$a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$$

We note that the factors are distinct. (Why ?)

(The reader is referred to Appendix VI B for the derivation of factors)

Case II : $b^2-4ac=0$

For this case, the real numbers r and s such that $r+s=b$ and $rs=ac$ are each $\frac{b}{2}$. (Why ?)

The quadratic polynomial can then be written as a product of two linear factors as

$$a\left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right) \qquad \text{(Why ?)}$$

We note that the factors are identical

In most cases, it will be easy to determine r and s (and consequently the factors) by trial and error. It is only when trial and error fails us that we will use the formula given above.

Example 1 : Factorize $3x^2+11x+9$

Solution : First, let us check if $3x^2+11x+9$ is factorizable over R.

$$b^2-4ac=(11)^2-4(3)(9)=13>0$$

Thus, indeed, $3x^2+11x+9$ is factorizable over R

Next, let us use trial and error, if possible, to determine r and s and split the term in x

We need r and s such that

$$r+s=11 \quad \text{and} \quad rs=27$$

None of the pairs of factors of 27, namely, 27,1 and 9, 3 satisfy the condition that their sum is 11 Trial and error, therefore, fails us. We use the formula given above and write

$$3x^2+11x+9=3\left[x+\left(\frac{11-\sqrt{13}}{6}\right)\right]\left[x+\left(\frac{11+\sqrt{13}}{6}\right)\right]$$

or,
$$3x^2+11x+9=\left(3x+\frac{11-\sqrt{13}}{2}\right)\left(x+\frac{11+\sqrt{13}}{6}\right)$$

Exercise 20.2

Factorize each of the following polynomials into linear factors, over R, if possible.

1. x^2-36
2. $-2x^2-3x+7$
3. $7x^2+11x+4$
4. $\frac{1}{2}x^2-3x+4$
5. $14x^2+9x+1$
6. $2x^2-3x+1$
7. $11\, b^2x^2+13\, bcx+2c^2,\ b \neq 0$
8. $3a^2x^2+abx+b^2,\ a \neq 0$
9. $4x^2-16x+15$
10. $rpx^2+(2qr-p^2)\, x-2pq,\quad r \neq 0,\ p \neq 0$
11. $\sqrt{2}x^2-3x-2\sqrt{2}$
12. $3x^2+4x+6$
13. $px^2+(4p^2-3q)\, x-12pq,\quad p \neq 0$
14. $3a^2x^2+2abx-b^2,\quad a \neq 0$
15. $y^2+6y+12$
16. $4\sqrt{3}x^2+5x-2\sqrt{3}$
17. $5x^2-4x+2$
18. Find the value(s) of 'k' for which x^2+kx+1
    (a) can be expressed,
    (b) cannot be expressed,
    as a product of linear factors, over R.

20.8. Quadratic Equations

Consider

$$x^2 = 0,\quad x^2-4 = 0,\quad \sqrt{2}x^2+3x+1 = 0,\quad x^2+7x = 0$$

These are examples of **second degree polynomial equations,** commonly known as **quadratic equations. In general $ax^2+bx+c = 0$, a, b, c $\in$ R, a $\neq$ 0 is called a quadratic equation over R.** The zeros of the quadratic polynomial ax^2+bx+c are called the **roots** or **solutions** of the quadratic equation $ax^2+bx+c = 0$

The notion of quadratic equations is, indeed, a very old one. Certainly, the Babylonians knew of quadratic equations some 4000 years ago. There are tablets, known as **Yale tablets**, dating back to 1600 B.C. which list scores of unsolved problems that involve quadratic equations.

The Greeks also made use of quadratic equations in solving geometrical problems. The well-known mathematician Euclid (born 365 B.C.), who created a famous geometry (Euclidian Geometry) gives, in his works, several problems that involve quadratic equations. In fact, he gives a clever solution to a quadratic equation in his book, called *Elements*. (See Vol. VI, Propositions 28 and 29)

The contributions of the Hindu mathematicians to quadratic equations are quite significant and extensive. It is said that the Hindus constructed 'altars' based on the solutions of the equation $ax^2+bx-c = 0$, as far back as the *Salva Sutra* period, dating back to roughly 500 B.C. Aryabhata (born A.D. 476) gives a rule to sum the geometric series which involves the solution of quadratic equations. Brahamagupta (born A.D. 598) provides a rule for the solution of the quadratic which is very much the quadratic formula. Mahavira, around A.D. 850, proposed a problem involving the use of a quadratic equa-

tion and its solution Sridhara, a Hindu mathematician, around A D. 1025 was the first to give the so-called **Hindu Rule** for the solution of a quadratic equation. This is what is now known as the **method of completing the squares.**

Al-khowarizmi, an Arab mathematician, in approximately A.D. 805 describes the two general methods of solving a quadratic equation Both these methods are based on the lines of work done by the Greeks

Another Arab, Omar Khayyam, around A.D 1100, also gives a rule for solving the quadratic equation.

The first important treatment of a quadratic equation. by factoring, is found in Harriot's works in approximately 1631. Others in the recent times who deserve special mention are Leonhard Euler (1707—1783), a Swiss mathematician ; E Bezout (1730—1783), a French mathematician and J.J. Sylvester (1814—1897), an English mathematician

20.9. Solutions of a Quadratic Equation : The Quadratic Formula

In Section 20.8, we have remarked that the problem of finding the roots of a quadratic equation $ax^2+bx+c = 0$ is equivalent to finding the zeros of the quadratic polynomial ax^2+bx+c. First, however, we state a very important theorem.

Theorem : **If a and b are any two real numbers such that ab = 0, then either a = 0 or b = 0 or both a = 0 and b = 0.**

(The reader is referred to Appendix VII for a proof of this theorem.)

Thus, given a quadratic equation if we can factor it into linear factors, we can set each factor equal to zero (by virtue of the above theorem) and obtain a solution

Example 1 : Find the roots of $x^2-5x+6 = 0$

Solution · The factors of x^2-5x+6 are $(x-3)$ and $(x-2)$. (Why ?)

Thus, $(x-3)(x-2) = 0$

Therefore, either $x-3 = 0$ i.e. $x = 3$

or $x-2 = 0$ i.e., $x = 2$

The roots of the equation are, thus, 2 and 3.

Let us perform a **check** ·

$x = 2$: Is $2^2-5(2)+6 = 0$? Yes, it is.

$x = 3$: Is $3^2-5(3)+6 = 0$? Yes, it is.

Example 2 : Find the solutions of $2x^2+x-6 = 0$

Solution :
$$2x^2+x-6 = 0$$
or,
$$2x^2+4x-3x-6 = 0$$
or,
$$2x(x+2)-3(x+2) = 0$$
or,
$$(x+2)(2x-3) = 0$$
Thus,
$$x = -2, \frac{3}{2}$$

(The reader is advised to perform a check.)

Example 3 : Find the solutions of $16x^2+24x+9 = 0$

Solution .
$$16x^2+24x+9 = 0$$

or, $$(4x+3)^2 = 0$$

i.e., $$x = \frac{-3}{4}, \; \frac{-3}{4}$$

(The reader is advised to perform a check.)

Example 4 : Find the roots of $3a^2x^2+8abx+4b^2 = 0, \; a \neq 0$

Solution: $$3a^2x^2+8abx+4b^2 = 0$$

or, $$3a^2x^2+6abx+2abx+4b^2 = 0$$

or, $$3ax(ax+2b)+2b(ax+2b) = 0$$

Thus $$x = \frac{-2b}{a}, \; \frac{-2b}{3a}$$

Let us perform a **check.**

$$x = \frac{-2b}{a} : \text{Is } 3a^2\left(\frac{-2b}{a}\right)^2+8ab\left(\frac{-2b}{a}\right)+4b^2 = 0\,?$$

i.e., Is $12b^2-16b^2+4b^2 = 0$? Yes it is.

$$x = \frac{-2b}{3a} : \text{Is } 3a^2\left(\frac{-2b}{3a}\right)^2+8ab\left(\frac{-2b}{3a}\right)+4b^2 = 0\,?$$

i.e. Is $\frac{4b^2}{3}-\frac{16b^2}{3}+4b^2 = 0$? Yes, it is

Example 5 : Find the roots of the equation $3y^2+2y+1 = 0$

Solution : The corresponding polynomial is $3y^2+2y+1$

We cannot find two real numbers r and s such that

$$r+s = 2$$

and $$rs = 3 \qquad \text{(Why ?)}$$

In fact, the polynomial cannot be factorized into linear factors over R.

(The reader is advised to calculate b^2-4ac and note that it does not satisfy the condition : $b^2-4ac \geqslant 0$)

Thus, the equation $3y^2+2y+1 = 0$ does not have any real roots. Could we have foretold this ? Let us study Example 6 below.

Example 6 : Find the roots of the quadratic equation

$$ax^2+bx+c = 0, \; a, b, c \in I^*, \; a \neq 0$$

Solution : $$ax^2+bx+c = 0$$

To be able to split the term in x and factorize is rather tedious. We give here an **alternative method** of finding the roots. This method is known as the **Hindu Rule** or the **Method of Completing the Squares.** We write

$$ax^2+bx = -c$$

or, $$x^2+\frac{b}{a}x = -\frac{c}{a} \qquad (1)$$

Now, what should we add to both sides of (1) to make the L.H.S. a perfect square ? The **rule** is to add ($\frac{1}{2}$ coefficient of x)^2. (Why ?) Thus, we obtain

*If a, b, c are rationals, we will always first multiply through by an appropriate integer to convert the rationals into integers

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

i.e.,
$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Whence,
$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Thus,
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The two roots of the quadratic equation $ax^2+bx+c=0$ are, therefore,

$$\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Now, let us concentrate on b^2-4ac. Since $a, b, c \in I$, $b^2-4ac \in I$. Trichotomy Axiom* tells us that b^2-4ac is either greater than zero or less than zero or equal to zero. Let us examine these possibilities one by one.

Case I : $b^2-4ac > 0$

If b^2-4ac is positive, its square root can be easily determined. In fact, if b^2-4ac is a perfect square, its square root is rational If b^2-4ac is not a perfect square. its square root is irrational.

In each case, the two roots are distinct. (Why ?)

Case II : $b^2-4ac = 0$

If b^2-4ac is zero, its square root is zero The two roots are, therefore, equal and rational.

Case III : $b^2-4ac < 0$

If b^2-4ac is negative, its square root cannot be determined in reals.

Thus, by simply calculating (b^2-4ac), we can comment on the nature of the roots of a quadratic equation. (b^2-4ac) is called the **discriminant** of the quadratic equation and is usually denoted by D.

We summarize the rule for telling the nature of the roots, given the value of the discriminant :

***Trichotomy Axiom** : Every real number a is such that either (*i*) $a > 0$, or (*ii*) $a = 0$ or (*iii*) $a < 0$.

Quadratic Equation : $ax^2+bx+c=0$, $a, b, c \in I$, $a \neq 0$
Discriminant $D = b^2-4ac$

Value of D	*Nature of the Roots*	*Roots are*
$D > 0$ and (i) a perfect square (ii) not a perfect square	Rational and Unequal Irrational and Unequal	$\dfrac{-b\pm\sqrt{D}}{2a}$
$D = 0$	Rational and Equal	$\dfrac{-b}{2a}$
$D < 0$	Not in reals	—

We conclude that the quadratic equation $ax^2+bx+c=0$, has roots in the reals if and only if $D \geqslant 0$

(In Example 5, $D=(2)^2-4(3)(1)=-8<0$)

Let us find the sum and the product of the roots.

Sum of the roots $= \dfrac{-b+\sqrt{b^2-4ac}}{2a} + \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

$= \dfrac{-b}{a}$

Thus, **the sum of the roots** $= \dfrac{-\mathbf{b}}{\mathbf{a}} = -\left(\dfrac{\textbf{coefficient of x}}{\textbf{coefficient of x}^2}\right)$

Product of the roots $= \left(\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right)$

$= \dfrac{(-b)^2-(\sqrt{b^2-4ac})^2}{(2a)^2}$

$= \dfrac{c}{a}$

Thus, **the product of the roots** $= \dfrac{\mathbf{c}}{\mathbf{a}} = \dfrac{\textbf{constant term}}{\textbf{coefficient of x}^2}$

We see from above that it is possible for us to find the sum and the product of the roots without actually determining the roots. For instance, in Example 1 of Section 20.9, we considered the quadratic equation $x^2-5x+6=0$. The roots were 2 and 3. Their sum is 5 and product is 6.

If we simply needed the sum and the product of the roots, we did not have to first calculate them. We could use the above formulae and obtain

$$\text{Sum} = \frac{-b}{a} = \frac{-(-5)}{1} = +5 \quad \text{and}$$

$$\text{Product} = \frac{c}{a} = \frac{6}{1} = 6$$

Exercise 20.3

1. Find the roots, if possible, of the following quadratic equations. Perform a check.

   (i) $x^2-x+\frac{1}{4} = 0$ (ii) $\frac{1}{4}x^2+\frac{5}{9}x+\frac{25}{81} = 0$

   (iii) $4x^2-9 = 0$ (iv) $25y^2+30y+9 = 0$

   (v) $6x^2-11x+3 = 0$

   (vi) $a^2p^2y^2-q^2 = 0$

   (vii) $ad^2x\left(\frac{a}{b}x+\frac{2c}{d}\right)+c^2b = 0$

   (viii) $x^2-7x+12 = 0$

   (ix) $10x^2+3bx+a^2-7ax-b^2 = 0$

   (x) $x^2-300 = 0$

   (xi) $r^2p^2z^2+2rpz+1 = 0$

2. Without determining the roots, comment upon their nature.

   (i) $2x^2+4x+3 = 0$ (ii) $x^2+x-1 = 0$

   (iii) $4y^2 = 1$ (iv) $6z^2-7z+2 = 0$

   (v) $\frac{4}{3}x^2-2x+\frac{3}{4} = 0$

3. Without determining the roots, write the sum and the product of the roots.

   (i) $6x^2-11x+3 = 0$ (ii) $4y^2 = 1$

   (iii) $\sqrt{3}y^2+11y+6\sqrt{3} = 0$

4. Find a quadratic equation whose roots are given below :

   (i) $2, \frac{1}{2}$ (ii) $\sqrt{3}, -\sqrt{3}$ (iii) $-3, 4$

20.10. Equations Reducible to Quadratic Form

At times, we come across equations which, upon suitable substitutions or simplifications, reduce to quadratic form. We give below some examples.

Example 1 : $\sqrt{y+1}+\sqrt{2y-5} = 3, \quad y \in R$

Solution : $\sqrt{y+1}+\sqrt{2y-5} = 3 \qquad (1)$

(1) can be written as

$$\sqrt{y+1} = 3 - \sqrt{2y-5} \tag{2}$$

Squaring both sides of (2), we obtain

$$y+1 = 9 - 6\sqrt{2y-5} + (2y-5)$$

or, $$y+3 = 6\sqrt{2y-5} \tag{3}$$

Squaring (3), we obtain

$$y^2+6y+9 = 36(2y-5)$$

or, $$y^2-66y+189 = 0 \tag{4}$$

We have thus reduced the given equation to a quadratic form (4) can be factored and we write

$$(y-63)(y-3) = 0$$

Thus, $$y = 63,\ 3$$

Let us perform a check.

y = 63 : Is $\sqrt{63+1}+\sqrt{126-5} = 3$? **No, it is not**

y = 3 : Is $\sqrt{3+1}+\sqrt{6-5} = 3$? **Yes, it is.**

We see that $y = 63$, although a solution of (4) does not satisfy the original equation $y = 63$ is called an **extraneous root**—extraneous because it is introduced either due to squaring the original equation or clearing fractions in the original equation

It is, therefore, necessary in problems of this type to always perform a check.

Example 2 : Find the solution of

$$\frac{1}{x+7}+\frac{1}{x+3} = \frac{6}{5},\quad x \in R,\ x \neq -7,\ x \neq -3$$

Solution :

$$\frac{1}{x+7}+\frac{1}{x+3} = \frac{6}{5} \tag{1}$$

Multiplying both sides of (1) by $5(x+7)(x+3)$, we obtain

$$5(2x+10) = 6(x+7)(x+3)$$

or, $$3x^2+25x+38 = 0$$

Whence, $$x = -2,\ -\frac{19}{3}$$

Check : x = −2 : Is $\frac{1}{5}+\frac{1}{1} = \frac{6}{5}$? **Yes, it is.**

x = $\frac{-19}{3}$: Is $\frac{3}{2}-\frac{3}{10} = \frac{6}{5}$? **Yes, it is.**

Example 3 : Solve $y(y+1)(y+3)(y+4)+2 = 0,\quad y \in R$

Solution : The above is a fourth degree equation in y. However, we can reduce it to a quadratic form as follows. We write

$$y(y+4)(y+1)(y+3)+2 = 0 \qquad \text{(Why ?)}$$

or, $$(y^2+4y)(y^2+4y+3)+2 = 0 \tag{1}$$

Let $y^2+4y = z$ (say) in (1) We obtain

$$z(z+3)+2 = 0$$

or, $$z^2+3z+2 = 0$$

or, $$(z+1)(z+2) = 0$$

Whence, $$z = -1, -2$$

Substituting the value of $z = y^2+4y$, we obtain

$$y^2+4y+1 = 0$$

Whence, $$y = -2 \pm \sqrt{3}$$

Also, $$y^2+4y+2 = 0$$

Whence, $$y = -2 \pm \sqrt{2}$$

(The reader is advised to perform a check)

20.11. Some Applications of Quadratic Equations

Example 1 : O girl ! out of a group of swans, $\frac{7}{2}$ times the square root of the number are playing on the shore of a tank. The two remaining ones are playing with amorous fight, in the water. What is the total number of swans ? (Bhaskara)

Solution : Let us denote the number of swans by x. Then, the number of swans playing on the shore of a tank $= \frac{7}{2}\sqrt{x}$

There are 2 remaining swans

$$\therefore \quad x = \frac{7}{2}\sqrt{x} + 2 \quad \text{or} \quad x-2 = \frac{7}{2}\sqrt{x}$$

or, $$x^2-4x+4 = \frac{49x}{4}$$

or, $$x^2-\frac{65x}{4}+4 = 0$$

Whence, $$x = 16, \tfrac{1}{4} \text{ (Why ?)}$$

$x = \frac{1}{4}$ is extraneous. In any case, we cannot have a quarter of a swan.

Thus, the number of swans is 16.

(The reader is advised to perform a check.)

Example 2 : Out of a certain number of Saras birds, one-fourth the number are moving about in lotus plants, 1/9th coupled with 1/4th as well as 7 times the square root of the number move on a hill ; 56 birds remain in Vakula trees What is the total number of birds ? (Mahavira)

Solution : Let us denote by x, the number of Saras birds.

Then, the number moving in lotus plants $= \frac{1}{4}x$

The number moving on the hill $= \frac{1}{9}x + \frac{1}{4}x + 7\sqrt{x}$

The number remaining $= 56$

Therefore, $$x = \frac{x}{4} + \frac{x}{9} + \frac{x}{4} + 7\sqrt{x} + 56$$

or, $$\frac{x}{18}-8 = \sqrt{x}$$

Thus, $$x^2-612x+(144)^2 = 0$$

Whence, $$x = 576, 36$$

36 is extraneous. The number of birds is, therefore, 576

(The reader is advised to perform a check.)

Example 3 : The sum of the squares of three consecutive natural numbers is 110. Determine the numbers.

Solution : Let the three consecutive natural numbers be

$$n, n+1, n+2$$

We are given that

$$n^2+(n+1)^2+(n+2)^2 = 110$$

Whence, $$n^2+2n-35 = 0$$

Thus, $$n = 5, -7$$

-7 is, obviously, inadmissible. The three natural numbers are, therefore, 5, 6 and 7.

(The reader is advised to perform a check.)

Exercise 20·4

Solve the following equations :

1. $\left(\frac{x-2}{x+2}\right)^2+3 = 4\left(\frac{x-2}{x+2}\right), \quad x \neq -2$
2. $\left(\frac{2x-3}{x-1}\right)-4\left(\frac{x-1}{2x-3}\right) = 3, \quad x \neq 1, x \neq \frac{3}{2}$

   (**Hint :** Let $\frac{2x-3}{x-1} = y$)
3. $\left(\frac{x}{x+1}\right)^2+6-5\left(\frac{x}{x+1}\right) = 0, \quad x \neq -1$
4. $\sqrt{2x^2-2x+1} = 2x-3$
5. $x-\sqrt{3x-6} = 2$
6. $\sqrt{11y-6}+\sqrt{y-1}-\sqrt{4y+5} = 0$
7. $9z^4+25 = 30z^2$

   (**Hint :** Let $z^2 = t$)
8. $x(x+3)(x+4)(x+7)+32 = 0$

*9. $2y^{-4}+5y^{-2} = 3, \quad y \neq 0$

   (**Hint :** Let $y^{-2} = z$)

10. The product of two consecutive integers is 3906. Determine the integers.

11 The sum of a number and its reciprocal is $\frac{13}{6}$. Determine the number.

12. The area of a right triangle is 30 sq. units Determine its base and the altitude, if the latter exceeds the former by 7 units

13. The length of a room is 3 m more than its breadth. If the area of the room is 70 sq. m, determine the dimensions of the room.

14 The product of two successive multiples of 3 is 180. Determine the multiples

15. A sum of Rs 100, at interest compounded annually, amounts to Rs 110.25 in two years. Determine the rate of interest.

16. On a pillar 9 cubits high is perched a peacock. From a distance of 27 cubits, a snake is coming to its hole at the bottom of the pillar. Seeing the snake the peacock pounces upon it. If their speeds are equal, tell me quickly, at what distance from the hole is the snake caught ? (Bhaskara)

17. Divide 51 into two parts whose product is 378.

20.12. Quadratic Inequations

In Chapter IV (See Part I of this book), we studied linear inequations. In this section, we shall concern ourselves with quadratic inequations in one variable.

The values of x, which satisfy a given inequation, are called the solutions of the inequation. There are two methods of finding the solutions

Method I : By factoring the corresponding polynomial This method is called the **algebraic method.**

Method II : By drawing the graph of the inequation. This method is called the **graphical method.**

Example 1 : Find the solutions of the inequation

$$x^2+3x-18 \geqslant 0$$

Solution :

Method I : The Algebraic Method

Consider $\qquad x^2+3x-18 \geqslant 0$

i.e., $\qquad (x-3)(x+6) \geqslant 0$

Now, for the product to be non-negative, two possibilities arise.

Case 1 : $\qquad (x-3) \geqslant 0$ and $(x+6) \geqslant 0$

i.e., $\qquad x \geqslant 3$ and $x \geqslant -6 \qquad (1)$

We note, in order that the two inequations in (1) are satisfied simultaneously, it is necessary that $x \geqslant 3$ is satisfied. (Why ?)

[The reader is advised to plot* $x \geqslant 3$ and $x \geqslant -6$ on the number line and obtain their intersection. (See Fig. 20 5)]

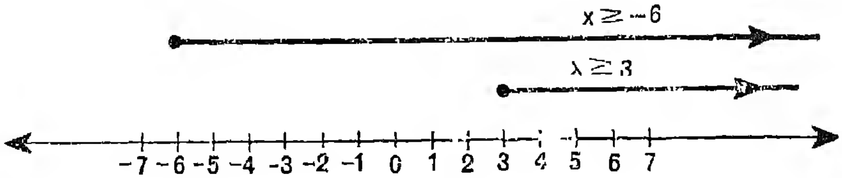


Fig. 20.5

The intersection is clearly $x \geqslant 3$

The solutions from Case 1 are, therefore, all those x's which satisfy $x \geqslant 3$.

Case 2 : $(x-3) \leqslant 0$ and $(x+6) \leqslant 0$

i.e., $x \leqslant 3$ and $x \leqslant -6$ (2)

It is easy to see that both the inequations in (2) are satisfied if $x \leqslant -6$.

The solutions from Case 2 are, therefore, all those x's which satisfy $x \leqslant -6$.

The two cases, together, give us all the solutions of $x^2+3x-18 \geqslant 0$. The solutions consist of all those x's for which $x \leqslant -6$ or $x \geqslant 3$.

We represent the solutions on the number line as follows. (See Fig. 20.6)

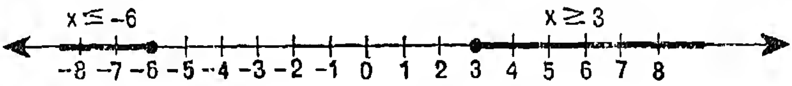


Fig. 20.6. Solutions of $x^2+3x-18 \geqslant 0$

Method II : The Graphical Method

The corresponding polynomial function is $f(x) = x^2+3x-18$. We have already learnt how to draw a graph of such a function. (See Fig. 20 7) We now want those values of x for which $x^2+3x-18 \geqslant 0$, i.e., $f(x) \geqslant 0$, i e., $y \geqslant 0$. In other words, we look for those values of x for which the curve is on or above the x-axis.

Clearly, these x-values are to the right of 3 (inclusive) or left of -6 (inclusive).

The solutions, therefore, consist of all those x's for which

$$x \leqslant -6 \quad \text{or} \quad x \geqslant 3$$

---

*The plot is a set of points on the number line However, in order to better show the intersection of the two plots, the plots have been raised above the x-axis

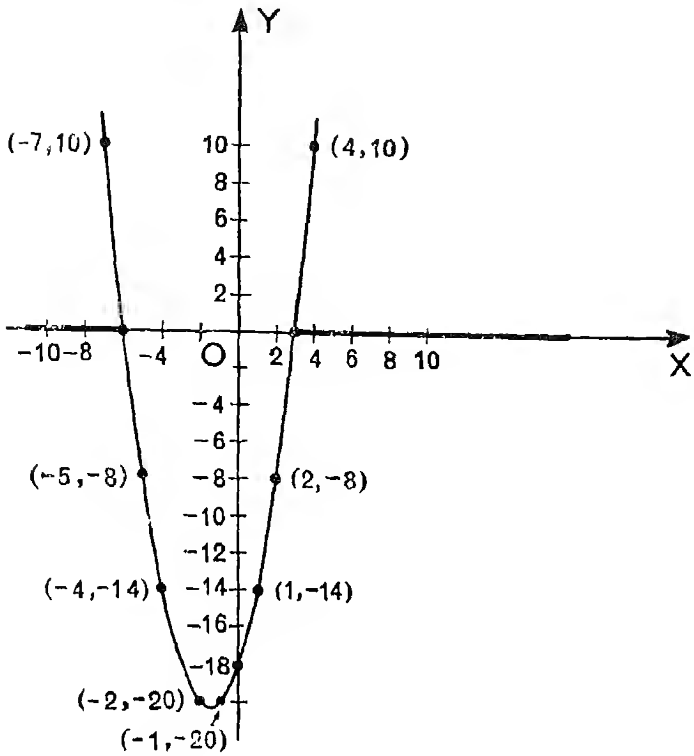


Fig. 20.7. Graph and solutions of $x^2+3x-18 \geqslant 0$

Example 2 : Find the solutions of the inequation

$$15x^2+4x-4 \leqslant 0$$

Solution :

Method I : Consider $15x^2+4x-4 \leqslant 0$

i.e., $(5x-2)(3x+2) \leqslant 0$ (Why ?)

Now for the product to be non-positive, two possibilities arise :

Case 1 : $(5x-2) \geqslant 0$ and $(3x+2) \leqslant 0$

i.e., $x \geqslant \frac{2}{5}$ and $x \leqslant -\frac{2}{3}$ (1)

There is, of course, no such value of x which satisfies both the inequations in (1). (Why?) Case 1, therefore, yields no solution(s).

Case 2 : $(5x-2) \leqslant 0$ and $(3x+2) \geqslant 0$

i.e., $x \leqslant \frac{2}{5}$ and $x \geqslant -\frac{2}{3}$

or, $$-\frac{2}{3} \leqslant x \leqslant \frac{2}{5}$$

The solutions, therefore, consist of all those x's for which

$$-\frac{2}{3} \leqslant x \leqslant \frac{2}{5}$$

We represent the solutions on the number line as follows. (See Fig. 20.8)

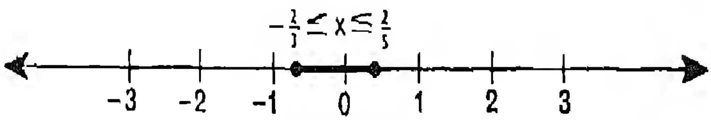


Fig. 20.8. Solutions of $15x^2+4x-4 \leqslant 0$

Method II · The reader is advised to draw a graph of the corresponding polynomial function

$$f(x) = 15x^2+4x-4$$

and look for those values of x for which $f(x) \leqslant 0$, i.e., $y \leqslant 0$ In other words, we look for those values of x for which the curve is on or below the x-axis

Exercise 20.5

1. Find the solutions of the following inequations. Use the graphical method.

   (*i*) $15x^2+4x-4 \geqslant 0$ (*ii*) $x^2-7x+6 > 0$

   (*iii*) $4-x^2 < 0$ (*iv*) $9-x^2 \geqslant 0$

   (*v*) $x^2-2x+1 \geqslant 0$

2. Find the solutions of the following inequations. Use the algebraic method. Represent the solutions on the number line.

   (*i*) $x^2-2x+1 < 0$ (*ii*) $x^2-4x-21 \geqslant 0$

   (*iii*) $2-3x-2x^2 \geqslant 0$ (*iv*) $x^2+x-12 \leqslant 0$

MISCELLANEOUS EXERCISE
(On Chapter XX)

1. Draw the graph of each of the following polynomials. Read the zero(s) from the graph
   (i) x^2-x-20 (ii) $4x^2+28x+13$
   (iii) $36-x^2$ (iv) x^2-5x+8
2. Factorize each of the following polynomials into linear factors, over R, if possible.
   (i) $\sqrt{3}y^2+10y+8\sqrt{3}$ (ii) $10x^2+153x+45$
   (iii) k^2+k+3 (iv) $-15x^2+7x+36$
   (v) $14x^2+18x+9$
3. Find the roots, if possible, of the following quadratic equations.
   (i) $4z^2-z-5=0$ (ii) $6x^2+40=31x$
   (iii) $3a^2x^2-abx=2b^2$ (iv) $48y^2-13y-1=0$
   (v) $28z^2-27z-36=0$ (vi) $6k^2+k(12+8a)+16a=0$
   (vii) $r^2s^2x^2+9t^2=-6rstx$
   (viii) $a^2b^2t^2-a^2t=1-b^2t$ (ix) $\sqrt{7}y^2-6y-13\sqrt{7}=0$
   (x) $a^2y^2-3aby+2b^2=0$
4. In the following equations, comment upon the nature of the roots, without calculating the roots
   (i) $a^2x^2+abx=b^2,\ a\neq 0$ (ii) $3x^2-6x+5=0$
   (iii) $5y^2+12y-9=0$
   (iv) $9a^2b^2x^2-48abcdx+64c^2d^2=0,\ a\neq 0,\ b\neq 0$
5. Without determining the roots, write the sum and the product of the roots.
   (i) $x^2+2x+4=0$ (ii) $\sqrt{3}ax^2=-10ax-7\sqrt{3},\quad a\neq 0$
   (iii) $28z^2-17z-56=0$ (iv) $x^2-2ax+a^2=0$
6. Find a quadratic equation whose roots are given below :
   (i) $\frac{1}{3},\ \frac{1}{2}$ (ii) $2\sqrt{3},\ -2\sqrt{3}$ (iii) $a,\ -2a$
   (iv) $\frac{4+\sqrt{7}}{2},\ \frac{4-\sqrt{7}}{2}$
7. Determine the value of k such that
   (i) $kx^2+4x+1=0$ has equal roots,

(*ii*) $x^2+5kx+16=0$ has no roots in reals.

(*iii*) $x^2+7(3+2k)-2x(1+3k)=0$ has equal roots.

8. Reduce the following equations to quadratic form and solve Check for extraneous roots.

(*i*) $(y^2-y)^2+5(y^2-y)+4=0$

(*ii*) $80x^{-2}=6x^{-1}+2,\quad x\neq 0$

(*iii*) $\left(\dfrac{7x-1}{x}\right)^2+3\left(\dfrac{7x-1}{x}\right)-18=0,\quad x\neq 0$

(*iv*) $x(x+5)(x+7)(x+12)+150=0$

(*v*) $6\left(\dfrac{y-3}{2y+1}\right)+1=5\sqrt{\dfrac{y-3}{2y+1}}\ ,\ y\neq -\frac{1}{2}$

$\left(\textbf{Hint}: \text{ Take } \sqrt{\dfrac{y-3}{2y+1}}=x\right)$

(*vi*) $\sqrt{y^2-y+2}+1=y$

9. In the interior of a forest, a number of apes equal to the square of $\frac{1}{8}$-th of their total number are playing with enthusiasm The remaining 12 apes are on hill. The echo of their shrieks by the surrounding hills rouses their fury. What is the total number of apes ? (Bhaskara)

10. In a tank beset with birds, the tip of a lotus is seen at a height of one palm $\left(\frac{1}{2}\text{-cubit}\right)$ Slowly propelled by the wind, it sinks in the water at a distance of two cubits. What is the depth of water in the tank ? (Bhaskara)

11. Surinder and Mohinder are two brothers Surinder's age is reciprocal of Mohinder's age and the sum of their ages is $\frac{10}{3}$ years. Determine their ages.

12. Solve the following inequations by the graphical method

(*i*) $4-(x-1)^2>0$ (*ii*) $2x^2-4x+1\leqslant 0$

(*iii*) $-x^2+2x-3\geqslant 0$ (*iv*) $-2x^2+9x+18\leqslant 0$

13. Solve the following inequations by the algebraic method. Represent the solutions on the number line.

(*i*) $x^2+x+\frac{1}{4}\geqslant 0$ (*ii*) $x^2-x-20\leqslant 0$

(*iii*) $(x-2)^2+1<0$

PERMUTATIONS, COMBINATIONS AND THE BINOMIAL THEOREM

21.1. The Fundamental Principle of Counting

Gautam wishes to distribute two toys between his two children What are the choices open to him and in how many ways can he distribute the toys ? Ravi goes to a movie The cinema hall has two entrances and three exits. In how many ways can Ravi enter and exit from the hall ? A coin is tossed two times and the outcomes are recorded. How many possible outcomes are there ? How many even 2-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 ? In how many ways can a school select a team of four players from among five 'good' badminton players to send to an inter-school tournament ? In how many ways can six examiners be selected from a list of ten ? We often come across problems of these types, in our daily life, which essentially involve the knowledge of **counting techniques** These problems are typical from among the class of problems that fall under the general heading of **Permutations and Combinations.**

Example 1 : A coin is tossed two times and the outcomes are recorded. How many possible outcomes are there ?

Solution : Let us denote by 'H' the coin turns up 'heads' and by 'T' the coin turns up 'tails'. Since the coin is tossed twice, we have the outcomes HH, HT, TH and TT. (TH, for instance, is the outcome where the coin turns up 'tails' on the first toss followed by 'heads' on the second toss.) Thus, we see that there are 4 possible outcomes when a coin is tossed twice.

Example 2 : Ravi goes to a movie. The cinema hall has two entrances and three exits. In how many ways can Ravi enter and exit from the hall ?

Solution : Let us denote the entrances by I and II and the exits by A, B and C. (See Fig. 21 1) The possible choices open to Ravi are :

IA, *IB*, *IC*, *IIA*, *IIB* and *IIC*

(*IC*, for instance, is the choice where Ravi enters from entrance *I* and exits from exit *C*.)

The number of ways Ravi can enter and exit from the hall is, therefore, 6.

The two examples above illustrate the use of a general principle called* the **Fundamental Principle of Counting.** The principle states :

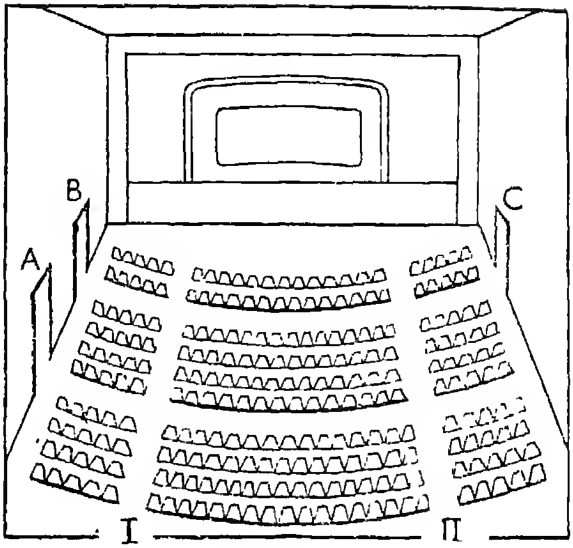


Fig. 21.1

If an event can occur in m different ways, following which, another event can occur in n different ways, then the total number of different ways of occurrence of both the events in the given order, is mn.** In other words, if an event can occur in **any one** of m different ways, and following the occurrence of this event, another event can occur in **any one** of n different ways, then the number of different ways of occurrence of both the events, in the specified order, is mn.

In Example 1, the coin can fall either 'heads' or 'tails' on the first toss. In other words, the number of possible outcomes on the first toss is 2. Corresponding to each outcome of the first toss, there are two possible outcomes on the second toss. The total number of different outcomes, therefore, is 2×2 or 4.

In Example 2, Ravi can enter from either entrance *I* or entrance *II*. Corresponding to each choice of entrance, Ravi has three choices for exit. Consequently, the number of ways in which Ravi can enter and exit is 2×3 or 6.

Example 3 : How many even 2-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 ?

Solution : Let us assume that the digits can be repeated. For a number to be even, the unit's digit, in this case, must be 2 or 4.

We, therefore, have 2 choices for the unit's place. Corresponding to each choice for the unit's place, we have a choice of any one of the five given digits for the ten's **place.** The total number of even 2-digit numbers is, therefore, 2×5 or 10.

The numbers are :

12, 14, 22, 24, 32, 34, 42, 44, 52, 54

*Some texts name this principle as the **Multiplication Principle.** We, however, prefer to call it the **Fundamental Principle of Counting.**

**The principle can easily be extended to more than two events.

(The reader is advised to determine the number of even 2-digit numbers in case repetition of digits is not allowed.)

It may not always be feasible to list the various possibilities and count. The Fundamental Principle is very helpful in such situations. Consider, for instance, the following example.

Example 4 : How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5, assuming that the digits cannot be repeated ?

Solution : In the unit's place, we have a choice of any one of five digits. Once this choice is made, we have only four digits remaining, any one of which can occupy the ten's place. Obviously, therefore, for the hundred's place, we have only 3 choices. We display this as follows :

$$\begin{matrix} 3 & 4 & 5 \\ h & t & u \end{matrix}$$

The three places, **in this order**, can, therefore, be filled in $5\times4\times3$ or 60 ways by the Fundamental Principle of Counting.

Thus, the number of 3-digit numbers from the given five digits is 60.

(The reader is advised to determine the number of 3-digit numbers in case the repetition of digits is allowed.)

Exercise 21.1

1. In how many ways can 3 women draw water from 3 taps, assuming no tap remains unused ?
2. How many 2-digit numbers can be formed from the digits 1, 2, 3, 4 and 5, assuming
   (*a*) repetition of digits is allowed ?
   (*b*) repetition of digits is not allowed ?
3. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5, assuming
   (*a*) repetition of digits is allowed ?
   (*b*) repetition of digits is not allowed ?
4. Determine the number of ways in which three books, one each in physics, chemistry and mathematics, can be arranged on a shelf.
5. Given a triangle with vertices A, B and C, in how many ways can the triangle be named ?
6. Find the number of ways in which four persons can line up in a queue for boarding a bus.
7. Four students compete in a race. In how many ways can the first two places be taken ?
8. How many different five letter words* can be formed out of the letters of the word 'KNIFE' assuming repetition of letters is not allowed?

*The words may or may not have a dictionary meaning.

9. Given four flags of different colours, how many different signals can be generated, if a signal requires the use of two flags, one below the other ?

*10. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there ? What if the coin is tossed four times ? five times ? Do you see a pattern in your answers ?
(The reader is advised to **invent** a formula for the number of outcomes of, say, n tosses of a coin.)

21.2. Permutations

Let us consider Example 1 again. In the two tosses of a coin, the outcome TH is different from the outcome HT. (Why ?) In other words, the order of occurrence of H and T is to be kept in mind. Similarly, in Example 3, for instance, 12 is a 2-digit even number. But if we interchange the order of occurrence of 1 and 2, we would no longer have a 2-digit even number.

Situations, in which the order of occurrence of events is important, give rise to what we call **permutations**.

A permutation is an arrangement, in a definite order, of a number of different objects. Each outcome HH, HT, TH, TT, therefore, in Example 1 is a permutation. Each choice of entrance followed by an exit, in Example 2, is a permutation. Each even 2-digit number, in Example 3, is a permutation. We give below some more examples of permutations.

Example 1 : Determine the number of permutations of three letters A, B and C. Write all the permutations.

Solution : Since the three letters have to be permuted or arranged and the order of occurrence of each letter in a particular arrangement is to be kept in mind, we can designate the place of occurrence of each letter as 1st, 2nd and 3rd place.

3	2	1
1st	2nd	3rd

Now in the 1st place, we can write either A or B or C. In other words, we have 3 choices for writing a letter in the 1st place.

For the 2nd place, we have a choice of one of the two remaining letters. (Why ?)

And for the 3rd place, we have only one choice of the letter that remains.

By the Fundamental Principle, therefore, the three places, in this order, can be filled in $3 \times 2 \times 1$ or 6 ways.

Let us write the six permutations. They are :

$$ABC,\ ACB,\ BCA,\ BAC,\ CAB \text{ and } CBA$$

(A triangle with vertices A, B and C, therefore, can be named in any one of the above six ways.)

We give here two important formulae for determining the number of permutations of a set of objects.

Proposition 1 : **The number nPn* of permutations of n different objects, taken all at a time, is $nPn = n(n-1)(n-2)...3.2.1 = n!$**

($n!$ is read as **'n factorial'** or **'factorial n'**. We note that $5! = 5.4.3.2.1 = 120$; $4! = 4.3.2.1 = 24$; $3! = 3.2.1 = 6$; $2! = 2.1 = 2$; $1! = 1$. We define $0! = 1$)

Proof : The number of ways of selecting the 1st object is n. Once the 1st object has been selected, there are $(n-1)$ that remain. Consequently the number of ways of selecting the 2nd object is $(n-1)$, and so on. Having selected $(n-1)$ objects in this way, there are $[n-(n-1)]$ or 1 that remain, which is the number of ways of selecting the nth object.

By the Fundamental Principle, therefore, the number of permutations of n different objects, taken all at a time, is the product

$$n(n-1)(n-2)...3.2.1$$

i.e.,
$$nPn = n!$$

Proposition 2 : **The number of permutations of n different objects, taken r at a time (nPr), $r \leqslant n$, is given by**

$$\mathbf{nPr = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}}$$

(The reader is advised to furnish a proof to Proposition 2 along the lines similar to the proof of Proposition 1.)

Obviously when $r = n$, Proposition 1 follows.

Example 2 : Rakesh, Hari, Karuna and Saleem, equally qualified, are to be appointed against four jobs in a factory. Determine the number of ways in which

(*i*) first and second job can be filled.

(*ii*) first three jobs can be filled.

(*iii*) all the four jobs can be filled.

Solution : (*i*) The first job can be filled by any one of the four persons. Having selected one of these four, there are only three that remain, any one of whom can be appointed against the second job.

The required number of ways, therefore, is 4×3 or 12.

(*ii*) By Proposition 2, the required number of ways is

$$4P3 = 4 \times 3 \times 2 \text{ or } 24$$

(*iii*) By Proposition 1, the required number of ways is

$$4P4 = 4 \times 3 \times 2 \times 1 \text{ or } 24$$

Example 3 : Gopal has 4 shirts, 4 pants and 2 handkerchiefs. Determine the number of ways in which Gopal can choose 1 shirt, 1 pant and 1 handkerchief.

Solution : The number of ways of choosing 1 shirt is, obviously, $4P1$, that of choosing 1 pant is $4P1$ and of choosing 1 handkerchief is $2P1$.

By the Fundamental Principle, therefore, the required number of ways is

$$(4P1)(4P1)(2P1) \text{ or } 4 \times 4 \times 2 = 32$$

Example 4 : Find the value of n such that

(*i*) $nP5 = 42(nP3),\ n > 4$

---

*The other notations, in use, are nP_n ; P^n_n, $P_{n,n}$; $P(n, n)$

$$(ii)\quad \frac{nP4}{(n-1)P4} = \frac{5}{3},\ n > 4$$

Solution : (*i*) $nP5 = 42(nP3)$

i.e., $n(n-1)(n-2)(n-3)(n-4) = 42n(n-1)(n-2)$

Dividing both sides by $n(n-1)(n-2)$, we obtain

$$(n-3)(n-4) = 42$$

i.e., $$n^2-7n-30 = 0$$

Whence, $$n = 10,\ -3$$

Obviously, n cannot be negative Thus $n = 10$ is the required value

(*ii*) $$\frac{nP4}{(n-1)P4} = \frac{5}{3}$$

i e, $3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)(n-3)(n-4)$

Dividing both sides by $(n-1)(n-2)(n-3)$, we obtain

$$3n = 5(n-4)$$

Whence, $$n = 10$$

Exercise 21.2

1. In how many ways can 1 male and 1 female player be selected for a mixed-doubles table-tennis match out of 2 male and 3 female players ?
2. How many 2-digit numbers can be formed from the digits 0 1, 2, 3 and 4, assuming repetition of digits
   (*a*) is allowed ?
   (*b*) is not allowed ?
3. Renu wants to arrange 3 economics, 2 history and 4 language books on a shelf. Find the number of arrangements, if all books on a subject are to be together.

*4. Prove that

$$n(n-1)(n-2)......(n-r+1) = \frac{n\,!}{(n-r)\,!}$$

[**Hint** : Multiply the numerator and denominator of the L.H.S. by $(n-r)\,!$]

5. Four science subjects and three languages are taught in a school. In how many ways can a student select one science subject and one language ?
6. Determine the number of different 5-letter words formed from the letters of the word 'EQUATION'.

*7. Find the value of r if

(*i*) $5(4Pr) = 6\,[5P(r-1)],\ r \geqslant 1$

(*ii*) $4(6Pr) = 6P(r+1)$

8. Determine the value of n if
   (i) $2(nP3) = (n+1)P3$
   (ii) $nP4 = 2(nP2)$
9. Prove that
$$n\,! = n(n-1)\,! = n(n-1)(n-2)\,! = n(n-1)(n-2)(n-3)\,!$$
10. In a restaurant, a menu lists 5 vegetables, 3 meats, 2 salads and 4 breads. In how many ways can a customer make a meal consisting of 1 meat, 1 vegetable, 1 salad and 1 bread ?

21.3. Combinations

Example 1 : How many line-segments can be formed by joining three non-collinear points A, B and C ?

Solution : Let us recall that a line-segment can be formed by joining any two points in a plane. Since A, B and C are non-collinear, the possible line-segments are . AB, BC and CA (See Fig 21.2) The number of line-segments is, therefore, 3

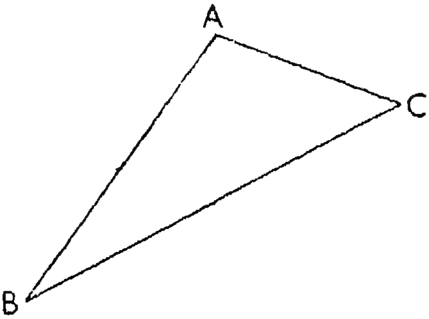


Fig. 21.2

Example 2 : A school has five 'good' badminton players A team of four has to be sent to an inter-school tournament. In how many ways can the team be selected ?

Solution : Let us denote by A, B, C, D and E the 5 'good' badminton players. The various teams of 4 that can be formed from among these players are :

$ABCD$, $ABCE$, $ABDE$, $ACDE$ and $BCDE$

Thus, the number of ways of selecting a team to be sent to the tournament is 5.

Situations in which a subset of a given set of objects is to be selected give rise to what we call **combinations.**

A combination is a selection of some or all of a number of different objects. In a combination, the order of selection of the objects is immaterial.

We give below a formula for determining the number of combinations of r objects, taken at a time, from a set of n objects.

Proposition 3 : The number of combinations of n different objects, taken r at a time (nCr*), $r \leqslant n$, is given by

$$nCr = \frac{n\,!}{r\,!\,(n-r)\,!}$$

(The proof of this proposition is considered outside the scope of this book.)

*The other notations, in use, are ${}^{n}C_{r}$; C^{n}_{r} ; Cn, r , $C(n, r)$, $\binom{n}{r}$

The number of line-segments formed by joining three non-collinear points is, therefore,

$$3C2 = \frac{3\,!}{2\,!\,(3-2)\,!} = 3$$

(See Example 1, Section 21.3)

Example 3 : Four friends Sudesh, Kamala, Rita and Moni want to cross a river in a boat The boat can carry only two passengers at a time. In how many ways can the friends pair off to cross the river ?

Solution : The four girls can pair-off as follows

Sudesh, Kamala ; Sudesh, Rita ; Sudesh, Moni , Kamla, Rita , Kamla, Moni ; and Rita, Moni

The number of ways in which the girls can pair off and cross the river is, therefore, 6.

Alternatively, using Proposition 3, the number of possible ways is

$$4C2 = \frac{4\,!}{2\,!\,2\,!} = 6$$

Example 4 : In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls ?

Solution : The number of ways of selecting 3 boys out of 5 is $5C3$ or 10 (Why ?)
The number of ways of selecting 3 girls out of 4 is $4C3$ or 4

By the Fundamental Principle, the required number of ways is, herefore, 10×4 or 40.

Example 5 : Give heuristic reasoning to show that

$$nCr = nC(n-r)$$

Solution : Every time we select r objects out of n, we leave behind $(n-r)$ objects. The number of ways of selecting r objects out of n is, therefore, exactly equal to the number of ways of leaving behind $(n-r)$ objects out of n. In other words,

$$nCr = nC\,(n-r)$$

(The reader is advised to prove the above algebraically by using the definition.)

It follows, therefore, that

$$4C3 = 4C1,\ 4C4 = 4C0,\ 5C1 = 5C4, \text{ etc.}$$

This relation is especially useful when large values of r are involved.

Exercise 21.3

1. From 5 consonants and 4 vowels, how many words can be constructed using 3 consonants and 2 vowels ?
2. Prove that $nC0 = 1$, n being a positive integer. Give an interpretation to this result in terms of selections of objects.
3. Find the number of combinations of the letters of the word 'UMESH', taken three at a time.

4. A contractor needs 2 carpenters Five, equally qualified, apply for the job. In how many ways can the contractor select the two ?
5. In how many ways can a committee of 3 be selected from 5 persons ?
6. How many words, each of three vowels and two consonants, can be formed from the letters of the word 'INVOLUTE' ?
7. In an examination, a student is to answer 4 questions out of 5. Questions 1 and 2 are, however, compulsory. Determine the number of ways in which the student can make the choice.
8. Verify : $4Cr+4C(r-1) = 5Cr$ for $1 \leqslant r \leqslant 4$
9. Solve the equation $nC2 = 10$ for n
10. A bag contains 4 black and 5 red balls 6 balls are drawn. Determine the number of ways in which 3 black and 3 red balls can be drawn.

21.4. When to use the formulae for permutations and when for combinations ? The reader would note that the words **'arrangement', 'ordering', 'permutation', 'arranging', 'line-up',** etc. are usually the key words in the problems that require the use of formulae for permutations. On the other hand, the words **'selection', 'subset', 'committee', 'selecting',** etc. are usually the key words in the problems that require the use of formulae for combinations.

The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject-matter as a self-contained topic in mathematics, under the name **Vikalpa.**

Before the advent of Jainism in India, in the Vedic period, we find the computations of the number of ways in which the poetic rhythms of verses can be altered.

In the 6th century B.C., Sushruta, in his medicinal work, *Sushruta Samhita,* asserts that 63 combinations can be made out of 6 different tastes (रस) taken one at a time, two at a time, etc. Pingala, a Sanskrit expert around 3rd century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his *Chhanda Sutra.* (छंद-सूत्र)

Among the Jains, Mahavira, of course, is the one who has made significant contributions to numerous branches of mathematics. He is the world's first mathematician credited with providing the general formulae for permutations and combinations.

Bhaskara (born A.D. 1114) treated the subject-matter of permutations and combinations under the name **Anka Pasha**, in his famous work *Lilavati*. In addition to the general formulae for nCr and nPr, already provided by Mahavira, Bhaskara gives several important theorems and results concerning the subject.

Outside India, the subject-matter of permutations and combinations had its humble beginnings in China in the famous book *I-King* (Book of Permutations). It is difficult to give the approximate time of this work, since in 213 B.C. the emperor had ordered all books or manuscripts in the country to be burnt which, fortunately, was not completely carried out.

Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations. Most of their work was a rediscovery of the work done in India

The first book which gives a complete treatment of the subject-matter of permutations and combinations is *Ars Conjectandi* written by a Swiss, Jakob Bernoulli (A D. 1654-1705), posthumously published in A.D. 1713.

21.5. The Binomial Theorem

The reader is already familiar with the concept of a 'binomial'. Let us see what we obtain if we raise* a binomial $(x+y)$ to a non-negative integral exponent.

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

We note from the above expansions that

1. **The number of terms in the expansion is always one more than the exponent.**
2. **In each expansion**
   (*a*) **the exponent of the 1st term (in x) is the same as the exponent of the binomial. (It follows, therefore, that the exponent of y in the 1st term is zero.)**
   (*b*) **Subsequently in each successive term, the exponent of x decreases by 1 with a simultaneous increase of 1 in the exponent of y,**
   (*c*) **until we arrive at the last term, in which the exponent of x is zero and that of y is equal to the exponent of the binomial.**
3. **In each expansion, the sum of the exponents of x and y in each term is equal to the exponent of the binomial.**
4. **The coefficients of various terms in an expansion form a pattern.** Let us examine the pattern :

Exponent of the binomial	*Coefficients of various terms*
0	1
1	1 — 1
2	1 — 2 — 1
3	1 — 3 — 3 — 1
4	1 — 4 — 6 — 4 — 1
5	1 5 10 10 5 1

*The reader is advised to verify the expansions by actual multiplications.

(*a*) **Each row of coefficients is bounded on both sides by 1.**

(*b*) **Any other coefficient, in a row, is the sum of the two coefficients, one on the immediate left and the other on the immediate right, in the immediately preceding row.** This is displayed by dotted markings forming triangles. For instance, in the row corresponding to the exponent 4, we have

$$4 = 1+3,\ 6 = 3+3,\ 4 = 3+1$$

These are bounded on both sides by 1.

The above arrangement of coefficients is now commonly known as **Pascal's Triangle.** Of French origin, Blaise Pascal (A.D. 1623—1662) constructed a triangle, in 1653, similar to the one above but written in a slightly modified form. He used the triangle to derive the coefficients of a binomial expansion. It was printed in 1665 and was known as **arithmetical triangle.**

The form in which we have used the triangle was known to the Italian, Nicola of Brescia, commonly known by the name Tartaglia (A.D. 1499—1557), the Germans Michael Stifel (A.D. 1486—1567) and Simon Stevin (A.D. 1548—1620).

The early Hindu mathematicians knew the coefficients in the expansion of $(x+y)^n$, $n \leqslant 7$ and a non-negative integer The arrangement of these coefficients was in the form of a diagram called **Meru-Prastara,** provided by Pingala in the third century B.C The triangular arrangement of Meru-Prastara is also found in the work of the Chinese mathematician, Chu Shi-Kie, in 1303. Pascal, therefore, was not the originator of the triangle. However, he highlighted several of the properties of the triangle and made extensive usage in his works, so much so that the name Pascal has come to be firmly attached to the triangle

The general expansion $(x+a)^n$, n a real number, is known by the name, **Binomial Theorem.** The study of binomial theorem is beyond the scope of this book We give below some examples of the binomial expansions and their applications.

Example 1 : Expand

(*i*) $(a+4)^5$

(*ii*) $(2-3x^2)^4$

Solution : (*i*) $(a+4)^5 = a^5(4)^0+5(a)^4(4)^1+10(a)^3(4)^2+10(a)^2(4)^3+5(a)^1(4)^4+(a)^0(4)^5$

Upon simplification, we obtain

$$(a+4)^5 = a^5+20a^4+160a^3+640a^2+1280a+1024$$

(*ii*) $(2-3x^2)^4 = (2)^4+4(2)^3(-3x^2)^1+6(2)^2(-3x^2)^2+4(2)^1(-3x^2)^3+(-3x^2)^4$

$= 16-96x^2+216x^4-216x^6+81x^8$

(The reader is advised to note how we employed the various coefficients from the Pascal's Triangle.)

Example 2 : Find the fourth term in the expansion of $\left(\frac{4}{7}x-q^2\right)^5$

Solution : From the Pascal's Triangle, the fourth term will have a coefficient 10.

Therefore,

$$\text{4th term} = 10\left(\frac{4}{7}x\right)^2(-q^2)^3$$

$$= \frac{-160}{49}x^2q^6$$

(The reader is advised to recall that **in a term of the expansion $(x+y)^5$, the exponent of y is one less than the term-number. The exponent of x can, then, easily be figured out since the sum of the exponents has to be 5.** Similar rule, of course, holds in other expansions.)

An alternate way to write the coefficients, in the Pascal's Triangle, is as follows :

Exponent of the binomial	*Coefficients of various terms*
0	1
1	$^1C_0$ $^1C_1$
2	$^2C_0$ $^2C_1$ $^2C_2$
3	$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$
4	$^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$
5	$^5C_0$ $^5C_1$ $^5C_2$ $^5C_3$ $^5C_4$ $^5C_5$

(The reader can easily verify the equivalence between the two triangles.)

Example 3 : Write the third term in the expansion of $\left(x+\frac{2}{5}y\right)^4$

Solution :

$$\text{3rd term} = (^4C_2)(x)^2\left(\frac{2}{5}\,y\right)^{3-1}$$

$$= \frac{24}{25}\,x^2y^2 \quad \text{(Why ?)}$$

Exercise 21.4

Expand the following :

1. $(x^2+y^2)^3$
2. $\left(x+\frac{1}{x}\right)^3$
3. $(2x-3y)^5$
4. $\left(-\frac{1}{2}x^2+\frac{1}{4}y\right)^4$
5. $\left(-3x-\frac{1}{3x}\right)^3$
6. Write the middle term in the expansion of $\left(x+\frac{1}{x}\right)^4$
7. Find the coefficient of x^3 in the expansion of $\left(\frac{x}{8}+y^2\right)^5$
8. Find the third term in the expansion of $\left(3x-\frac{y^3}{6}\right)_4$

MISCELLANEOUS EXERCISE
(On Chapter XXI)

*1. A die is tossed once and the outcome is recorded. How many possible outcomes are there ? What if the die is tossed two times ? Three times ? Do you see a pattern in your answers ?

(The reader is advised to **invent** a formula for the number of outcomes of, say, n tosses of a die.)

2. How many 2-digit numbers can be formed from the digits 8, 1, 5, 3 and 4 assuming

(*a*) repetition of digits is allowed ?

(*b*) repetition of digits is not allowed ?

*3. How many numbers below 1000 can be formed from the digits 2, 8, 5, 6 and 9 if

(*i*) no digit is repeated ?

(*ii*) repetition of a digit is allowed ?

4. How many different 5-letter words can be formed out of the letters of the word 'DELHI' ? How many of these will begin with D and end with I ?

*5. How many different numbers can be formed from the digits 1, 2, 3, 5 and 7 ? Assume that the digits cannot be repeated. How many of these will be even ?

6. In how many ways can two friends sit in three vacant seats in a bus ?

7. In how many ways can Indira and Pindi join four colleges if both do not join the same college ?

8. Determine the number of permutations of the letters of the word 'HEXAGON' taken all at a time.

9. Four tourists have a choice of 5 hotels in a city. Each wants to stay in a different hotel. Find the number of ways in which they can make the choice.

10. Five songs are to be given in a variety entertainment programme. In how many different orders could they be presented ?

11. Find the number of ways in which 5 different coins, when tossed simultaneouly, will turn up 2 heads and 3 tails.

12. Three points are taken on a circle. How many chords can be drawn by joining the points in all possible ways ? What if four points are taken on the circle ? Five points ?

13. Given 4 points in a plane, no three of which are collinear, determine the number of lines that can be drawn by joining these points. What if we have 5 points in the plane ? 6 points ?
14 How many 2-letter words can be formed using the letters of the word 'ZEBRA' ? How many of these will not contain vowels ?
15. Verify :
    (*i*) $2C0+2C1+2C2 = 2^2$
    (*ii*) $3C0+3C1+3C2+3C3 = 2^3$
    (*iii*) $4C0+4C1+4C2+4C3+4C4 = 2^4$

    What will be the value of $5C0+5C1+5C2+5C3+5C4+5C5$?
16 (*a*) Prove that the number of **all** subsets of a set consisting of two elements is 4
    (*b*) What will be the number of all subsets of a set consisting of 3 elements ? 4 elements ? n elements ?

    (**Hint :** Use Question 15)
17. A bag contains 5 red, 4 blue and 3 white marbles. In how many ways can a person select 2 red, 4 blue and 1 white marble ?
18. In how many ways can 5 objects be divided among A, B and C if A must receive 2, B must receive 1 and C must receive 2 objects ?
19. Solve : $nC4 = nC6$ for n .
*20. Prove . $nPr = r!\,(nCr)$. Give an argument in terms of arrangements and selections of objects justifying the above result.
21. How many diagonals does a pentagon have ?
22. Expand the following :
    (*a*) $(a-2b)^3$
    (*b*) $\left(\frac{2}{3}t - \frac{3}{2t}\right)^4$
    (*c*) $(4x-5y)^5$
    (*d*) $(y^2+3x)^3$
23. Find the 4th term of $(-3a-b)^5$
24. Determine the coefficient of z^6 in the expansion of $(3x^3-2z^2)^4$
25. Determine the two middle terms in the expansion of $(x^2+a^2)^8$

CHAPTER XXII

CIRCLES

22.1. Introduction

So far, we have studied the geometry of plane figures which are composed of straight lines/line-segments, such as angles, triangles, rectangles, polygons, etc. However, we also come across several plane figures which are not composed of straight lines/line-segments. The simplest among these is a circle

A CIRCLE is a closed curve in a plane, each of whose points is at the same (constant) distance from a given point in the plane. The given point is called the CENTRE of the circle and the constant distance the RADIUS

The segment joining the centre to any point on the circle is also called a **radius** of the circle. In Fig 22.1, C is a circle with centre O and P is any point on the circle. OP is a **radius** of the circle. It is obvious that if Q is any point **outside** the circle C, $OQ > OP$ and if R is any point **inside** the circle C, $OR < OP$.

Any segment whose end-points are on a circle is called a **chord** of the circle. In Fig. 22.2, AB is a chord of the circle C.

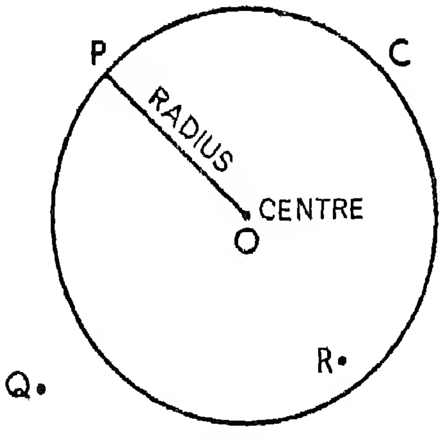


Fig. 22.1

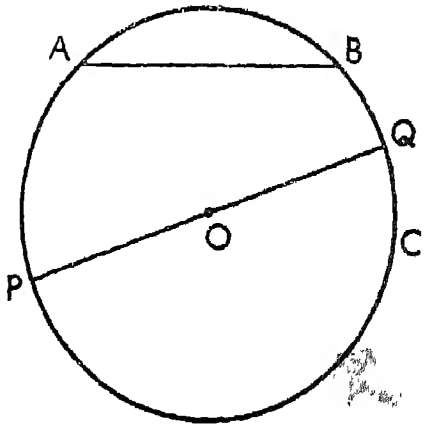


Fig. 22.2

Any chord of a circle containing the centre is called a **diameter** of the circle. In Fig. 22.2, PQ contains the centre O and, therefore, is a **diameter** of the circle.

When we need to refer to the length of a diameter or of a radius, we will use the term 'diameter' or 'radius', as the case may be, to denote the length. However, this will not cause any confusion since the context will always make it clear whether we are referring to the diameter (or radius) or its length. We note that **the diameter d of a circle is twice its radius r, i.e.,**

$$\mathbf{d = 2r}$$

The part of the plane enclosed by a circle is called a **circular region**.

22 2 Circles Through Given Points

Let P be any given point in a plane. Let O be another point in this plane. With O as centre and OP as radius, we can draw a circle that passes through P Now, **given the point P, how many circles can we draw that pass through P ?** Obviously, as many as we like, since the other point O can be taken anywhere in the plane of the point P.

Next, let P and Q be any two given points in a plane. **How many circles can we draw that pass through both P and Q ?** For a circle to pass through P and Q, it is necessary that its centre must be equidistant from both P and Q In other words, its centre must lie on the perpendicular bisector of the segment PQ Therefore, again we can draw as many circles as we like passing through two given points in a plane

How about through three given points in a plane ?

Theorem 14 · **Through three given non-collinear points in a plane, there passes one and only one circle.**

Given : Three non-collinear points A, B and C. (See Fig. 22 3)

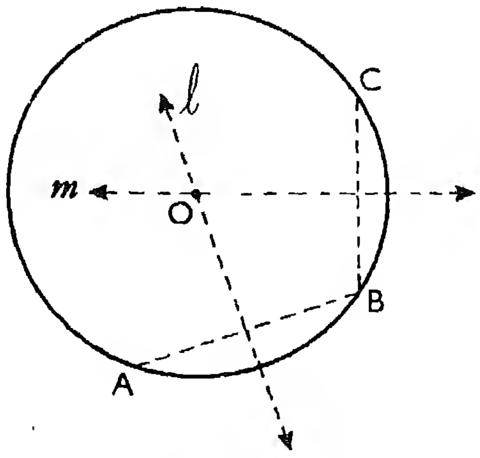


Fig. 22.3

To prove : One and only one circle passes through A, B, and C. In other words,

(*i*) One circle can pass through A, B and C,

(*ii*) Only one circle can pass through A, B and C.

Construction : Join A and B as also B and C Draw the perpendicular bisectors l and m of the segments AB and BC, respectively.

Proof : Since A, B and C are three non-collinear points, AB and BC are not in the same line.

Therefore, l and m, their perpendicular bisectors, are not parallel. Let them intersect at the point O.

Now, l is the perpendicular bisector of AB. Therefore, every point on l is equidistant from both A and B.

Similarly, every point on m is equidistant from both B and C.

But O lies on both l and m. Therefore,

$$OA = OB = OC \qquad \text{(Why ?)}$$

Hence, a circle with centre O and radius OA can be drawn which will pass through A, B and C.

Thus, one circle can pass through A, B and C.

Q.E.D. (i)

It is necessary that for any circle to pass through A, B and C, its centre must lie on the perpendicular bisectors, l and m, of AB and BC, respectively.

But O is the only point which lies on both l and m. (Why ?)

Therefore, only one circle can pass through A, B and C.

Q.E.D. (ii)

What if the three points A, B and C were collinear ? We note, in this case, that the perpendicular bisectors l and m would be parallel, i.e., will not intersect. Thus, it will not be possible to find a point equidistant from A, B and C.

Hence, if the three given points are collinear, no circle can be drawn to pass through them.

Now, what if we are given four points in a plane ? How many circles can be drawn to pass through them ? We will only assert that the points have to be suitably situated for the circle to pass through four points. The conditions under which such a circle can be drawn will be discussed elsewhere in this chapter.

Circumcircle of a Triangle : **A circle passing through the three vertices of a triangle is called the CIRCUMCIRCLE of the triangle. The centre of the circumcircle is called the CIRCUMCENTRE of the triangle.**

Given a triangle ABC, how do we draw its circumcircle ? Let us refer to Fig. 22.4. We draw the perpendicular bisectors of any of its two sides, say, l of the side AB and m of the side BC.

Denote the point of intersection of l and m by O. With O as centre and OA (or OB or OC) as radius, we draw a circle.

This circle is the circumcircle and its centre O is the circumcentre of $\triangle ABC$.

What if we were also to draw the perpendicular bisector of the third side AC? Since $OA = OC$, we would find that O would also lie on the perpendicular bisector of AC

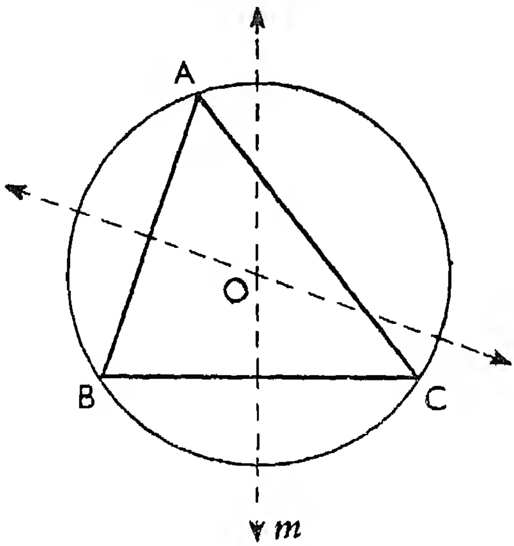


Fig. 22.4

(The reader is advised to verify for himself that the perpendicular bisector of AC also passes through O.)

Thus, we observe that

The perpendicular bisectors of the three sides of a triangle are concurrent.

22.3. Chord and the Perpendicular from the Centre

Theorem 15 : **The perpendicular from the centre of a circle to a chord bisects the chord.**

Given : A circle with centre O. AB is a chord. OD is the perpendicular from the centre to the chord AB, D being the foot of the perpendicular. (See Fig. 22·5)

To prove : $AD = BD$

Construction : Join O and A as also O and B.

Proof : Since $OD \perp AB$, OAD and OBD are right triangles.

Now, in $\triangle$s OAD and OBD,

$$OA = OB \qquad \text{(Radii of the same circle)}$$
$$OD = OD$$

Therefore, $\triangle OAD \cong \triangle OBD$ $\qquad$ (RHS)

Whence, $AD = BD$

Q.E.D.

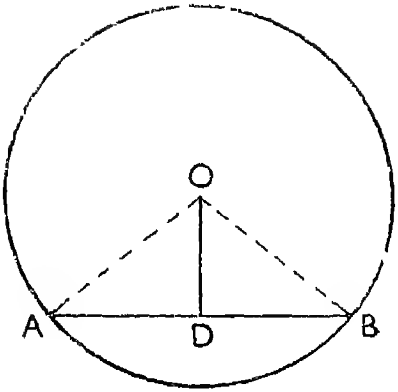


Fig. 22.5

Converse of Theorem 15 : **The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.**

Given . A circle with centre O AB is a chord. D is the mid-point of AB (See Fig 22.6)

To prove : $OD \perp AB$

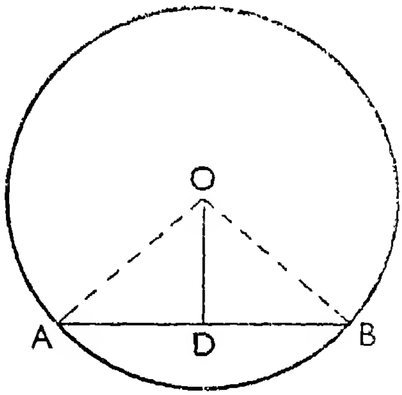


Fig. 22.6

Construction : Join O and A as also O and B.

Proof : In $\triangle$s OAD and OBD,

	$OA = OB$	(Radii of the same circle)
	$AD = BD$	(Given)
	$OD = OD$	
Therefore,	$\triangle OAD \cong \triangle OBD$	(SSS)

Whence, $\angle ADO = \angle BDO$

But, $\angle ADO + \angle BDO = 180°$

Therefore, $\angle ADO = 90°$

i.e., $OD \perp AB$

Q.E.D.

22.4. Symmetries of a Circle

Let us consider a circle with diameter AB. (See Fig 22.7) Let P be any point on the circle.

If we draw PQ perpendicular to AB meeting the circle at Q, the mid-point L of PQ will be on the diameter AB. (Why ?)

Now, AB is the perpendicular bisector of the chord PQ Therefore, Q is the image of the point P with respect to the diameter AB. In other words, the image of the point P with respect to the diameter AB also lies on the (same) circle.

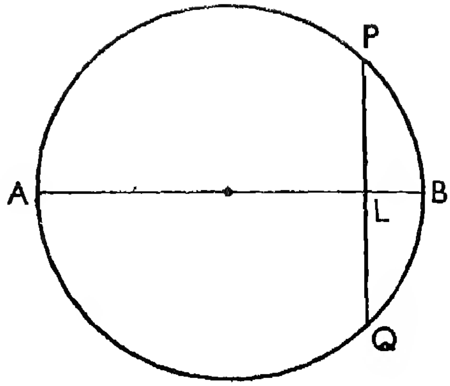


Fig. 22.7

The image of every point on the circle with respect to the diameter AB, therefore, also lies on the (same) circle. In other words, if the circle is folded at the diameter AB, the part of the circle on one side of the diameter will coincide exactly with the part of the circle on the other side. It follows that the diameter AB is an axis of symmetry of the circle. **Every diameter, therefore, is an axis of symmetry of the circle.** (See also Chapter XIII, Section 13.3 of Part I of this book.)

Example 1 : The length of a chord of a circle of radius 10 cm is 12 cm. Determine the distance of the chord from the centre.

Solution : Let C be a circle of radius 10 cm with centre O (See Fig. 22.8) Let PQ be a chord of length 12 cm. Draw $OM \perp PQ$. Then,

$$PM = MQ = \frac{1}{2}PQ = 6 \text{ cm} \quad \text{(Why ?)}$$

Now, OMP is a right triangle.

Therefore, $OM^2 = OP^2 - PM^2 = 64$

(Pythagoras Theorem)

$\therefore \quad OM = 8$ cm

i.e., the distance of the chord from the centre is 8 cm.

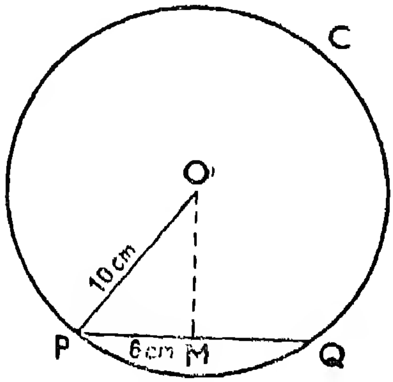


Fig. 22.8

Example 2 : Prove that the line joining the mid-points of the two parallel chords of a circle passes through the centre.

Solution : Let AB and CD be the two parallel chords of a circle with centre O and let M and N be their mid-points, respectively. (See Fig. 22.9)

We need to show that the line joining M and N passes through the centre O, i.e., O lies on MN. Let us join O and M as also O and N. We note that

line $OM \perp AB$.

But, $AB \parallel CD$

Therefore, line $OM \perp CD$ (*i*)

Also, line $ON \perp CD$ (Why ?) (*ii*)

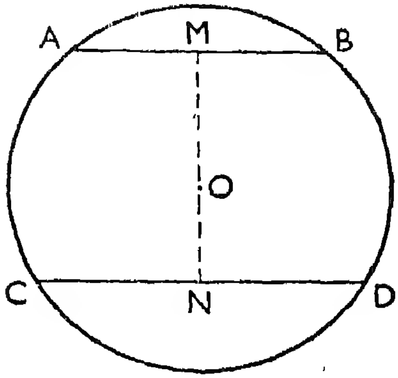


Fig. 22.9

From (*i*) and (*ii*), it follows that OM and ON are two perpendiculars, through O, to the same chord CD.

Therefore, OM and ON must be the same line, i.e., O lies on MN.

Exercise 22.1

1. Draw the circumcircle of a $\triangle ABC$ given that $AB = 6$ cm, $BC = 5$ cm and $CA = 3$ cm.
2. A chord of a circle of radius 13 cm is at a distance of 5 cm from the centre. Determine the length of the chord.
3. Determine the radius of a circle, one of whose chords, of length 8 cm, is at a distance of 3 cm from the centre.
4. Given a circle of radius 5 cm, AB and CD are its two parallel chords of lengths 8 cm and 6 cm respectively. Determine the distance between them if the two chords are on (*i*) the same side, (*ii*) opposite sides of the centre.
5. Given a circle of radius $\mathbf{r}$, PQ and PR are its two chords such that $PQ = 2PR$. If PQ and PR are at a distance $\mathbf{a}$ and $\mathbf{b}$ respectively from the centre, prove that

$$4\mathbf{b}^2 = \mathbf{a}^2 + 3\mathbf{r}^2$$

6. In Fig 22.10, two circles with centres O and P respectively intersect each other at the points A and B. Show that OP is the perpendicular bisector of the common chord AB.

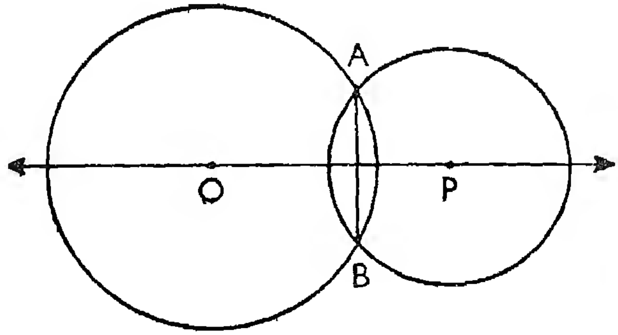


Fig. 22.10

7. In Fig. 22.11, l is a line intersecting the two circles, whose common centre is O, at the points A, B, C and D. Show that $AB = CD$.

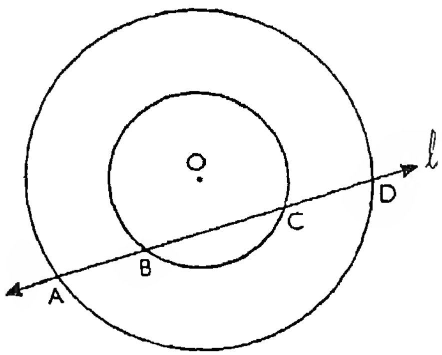


Fig. 22.11

Note : **Circles, having the same centre, are called CONCENTRIC CIRCLES.**

8. In Fig. 22.12, ABC is a triangle in which $AB = AC$. Prove that the bisector of $\angle BAC$ passes through the centre of the circumcircle of $\triangle ABC$.

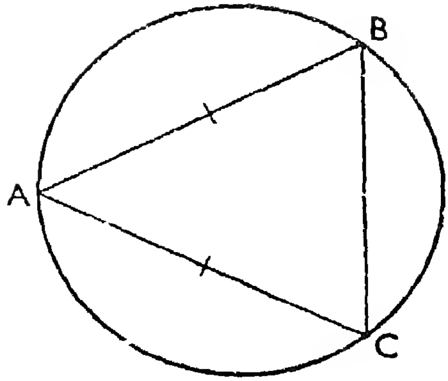


Fig. 22.12

9. In Fig. 22.13, PQ, a diameter of the circle, bisects the two chords AB and CD at the points M and N respectively. Prove that $AB \parallel CD$.

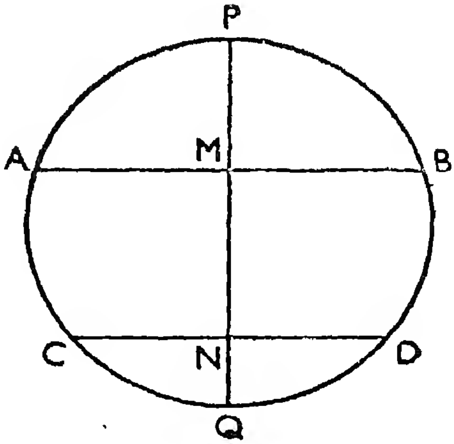


Fig. 22.13

10. Prove that if two parallel chords of a circle are each bisected by a third chord, the third chord is a diameter of the circle.

11. Determine the locus of the mid-points of the parallel chords of a circle.

12. In Fig 22 14, AB and BC are two chords of a circle whose centre is O such that $\angle ABO = \angle CBO$ Show that $AB = CB$.

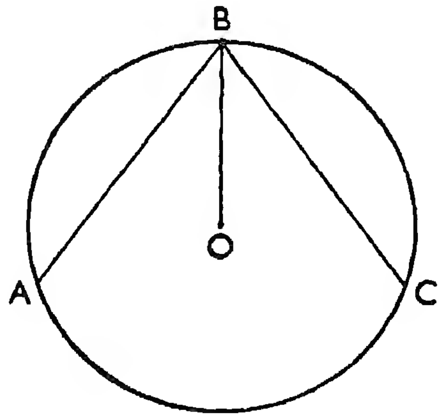


Fig. 22.14

13. In Fig. 22.15, OD is perpendicular to the chord AB of a circle, whose centre is O. If BC is a diameter, show that $CA = 2\ OD$.

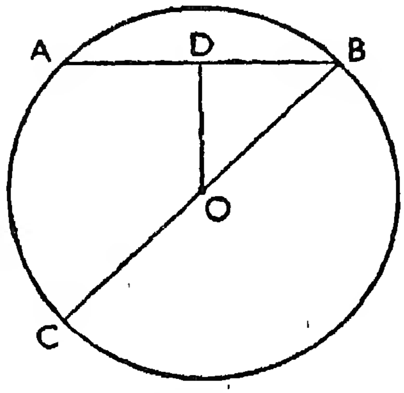


Fig. 22.15

14. Prove that the perpendicular bisector of a chord of a circle passes through its centre.

22.5. Chords Equidistant from the Centre

Theorem 16. **Equal chords of a circle are equidistant from the centre.**

Given. AB and CD are equal chords of a circle whose centre is O
$OM \perp AB$ and $ON \perp CD$. (See Fig 22 16)

To prove $OM = ON$

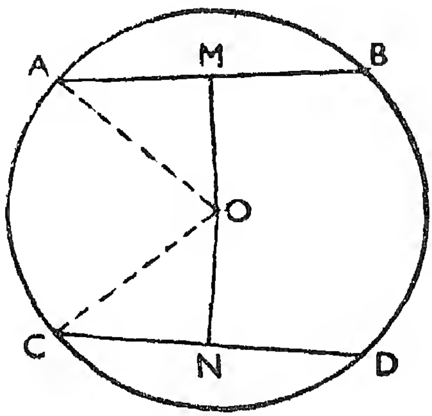


Fig. 22.16

Construction : Join O and A as also O and C.

Proof : $AB = CD$ (Given)

$\therefore$ $AM = CN$ (Theorem 15) (i)

Now, in right triangles OMA and ONC,

$OA = OC$ (Radii of a circle)

$AM = CN$ [Proved in (i) above]

Therefore, $\triangle OMA \cong \triangle ONC$ (Why ?)

Whence, $OM = ON$

Q.E.D.

Converse of Theorem 16 : **Chords of a circle equidistant from the centre are equal.**

Given : AB and CD be two chords of a circle whose centre is O. If $OM \perp AB$ and $ON \perp CD$, then $OM = ON$. (See Fig. 22.17)

To prove : $AB = CD$

Proof : In right triangles OMA and ONC

$OA = OC$ (Radii of a circle)

$OM = ON$ (Given)

Therefore, $\triangle OMA \cong \triangle ONC$ (Why ?)

Whence, $AM = CN$

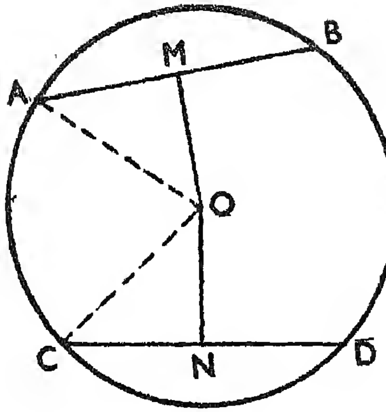


Fig. 22.17

Thus, $2\,AM = 2\,CN$

i.e., $AB = CD$ **Q.E.D.**

(The reader is advised to furnish a proof to Theorem 16 and its converse by using Pythagoras Theorem. The proofs will be similar to Example 2 that follows.)

Example 1 : If two equal chords of a circle intersect each other, show that the respective parts of a chord are equal to the corresponding parts of the other.

Solution : Let AB and CD be two equal chords of a circle whose centre is O. Let the chords intersect at the point P. (See Fig. 22·18)

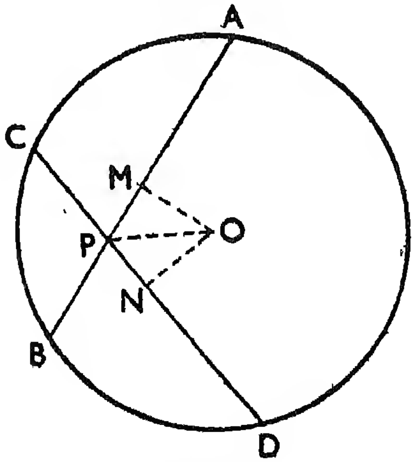


Fig. 22.18

We need to show that

(*a*) $AP = DP$ and

(*b*) $BP = CP$

Let us draw $OM \perp AB$ and $ON \perp CD$. Further, let us join O and P.

Now, since $AB = CD$, it follows that

$OM = ON$ (Theorem 16)

Thus, $\triangle OMP \cong \triangle ONP$ (*RHS*)

Whence, $MP = NP$ (*i*)

Also, $AM = DN$ (Theorem 15) (*ii*)

Adding (*i*) and (*ii*), we obtain

$AP = DP$ (*iii*)

But $AB = CD$ (Given) (*iv*)

Subtracting (*iii*) from (*iv*), we obtain

$BP = CP$

Example 2 : Of the two chords AB and CD of a circle, AB is nearer to the centre O than the chord CD. Prove that $AB > CD$.

Solution : Let us denote the radius of the circle by r. (See Fig. 22.19)

Let us draw $OM \perp AB$ and $ON \perp CD$.

Since AB is nearer to O than CD, it is obvious that

$$OM < ON$$

We need to show that

$$AB > CD$$

Join O and A as also O and C.

Now, $AM^2 = r^2 - OM^2$ (*i*)

and, $CN^2 = r^2 - ON^2$ (*ii*)

But, $OM < ON$ (Given)

$\therefore$ $r^2 - OM^2 > r^2 - ON^2$ (Why ?)

Whence, $AM^2 > CN^2$

i.e., $AM > CN$

Whence, $AB > CD$ (Since $AB = 2\ AM$, $CD = 2\ CN$)

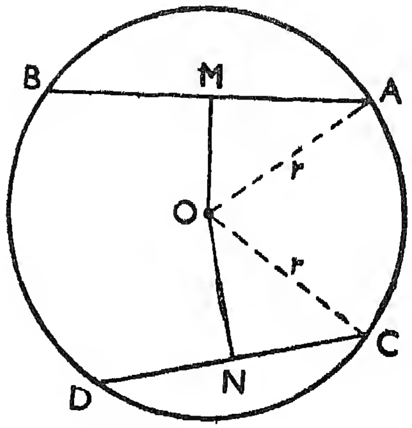


Fig. 22.19

Exercise 22.2

1. In Fig. 22.20, AB and AC are equal chords of a circle whose centre is O.

   Show that ray AO is the bisector of $\angle BAC$.

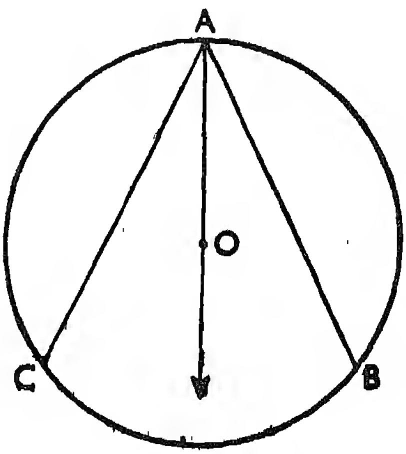


Fig. 22.20

2. In Fig 22.21, AB and CD are parallel chords of a circle whose diameter is AC.

   Prove that $AB = CD$.

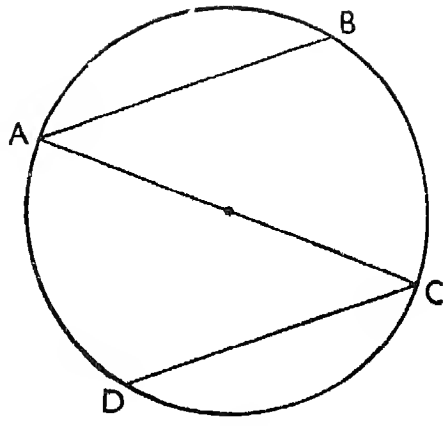


Fig. 22.21

3. Prove Theorem 16 and its converse by using Pythagoras Theorem.
4. In Fig. 22 22, AB and CD are equal chords of two congruent* circles with centres O and P. If $OM \perp AB$ and $PN \perp CD$, prove that $OM = PN$.

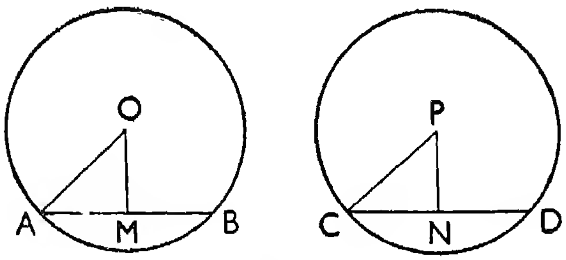


Fig. 22.22

(The above result can also be stated as :

Equal chords of congruent circles are at the same distance from the respective centres.)

5. State and prove the converse of the above statement.

*The reader is already familiar with congruent figures (Chapter XIII). It is obvious that congruent circles have equal radii.

6. In Fig. 22·23, AB and CD are chords of a circle whose centre is O. The two chords intersect at P. If PO bisects $\angle APD$, prove that $AB = CD$.

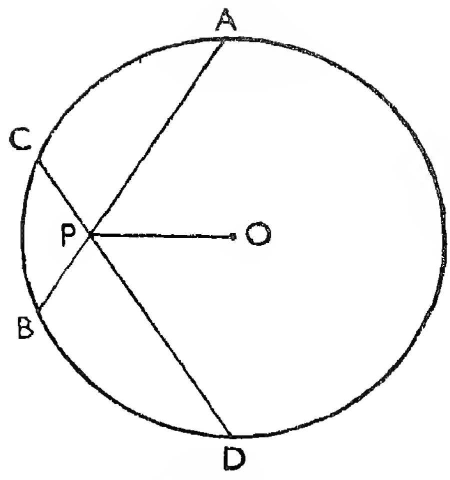


Fig. 22.23

7 If two chords of a circle are unequal, prove that the greater chord is nearer to the centre.

(**Note :** This is the converse of Example 2, Section 22 5)

8. In Fig. 22 24, AB and CD are equal chords of a circle whose centre is O. If M is the mid-point of AB and N is the mid-point of CD, prove that

$$\angle AMN = \angle CNM$$

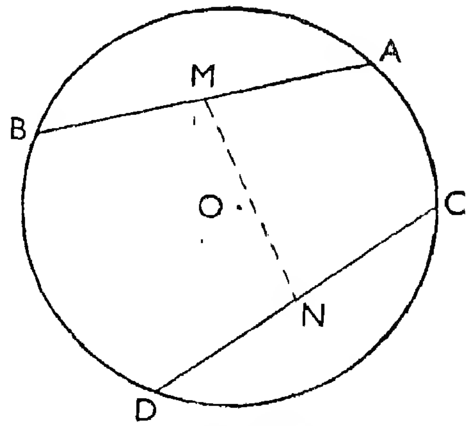


Fig. 22.24

9. In Fig. 22.25, AB and CD are equal chords of a circle whose centre is O When produced, these chords meet at E.

Prove that $EB = ED$ and $EA = EC$.

(**Hint** : Draw perpendiculars from the centre O to the two chords AB and CD.)

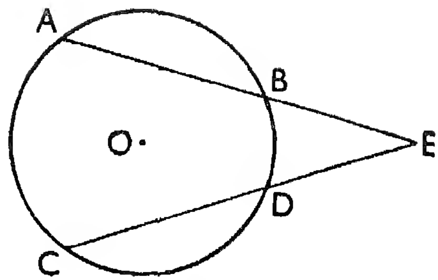


Fig. 22.25

10. In Fig. 22.26, AB and CD are equal chords of a circle whose centre is O. $OM \perp AB$ and $ON \perp CD$.

Prove that $\angle OMN = \angle ONM$.

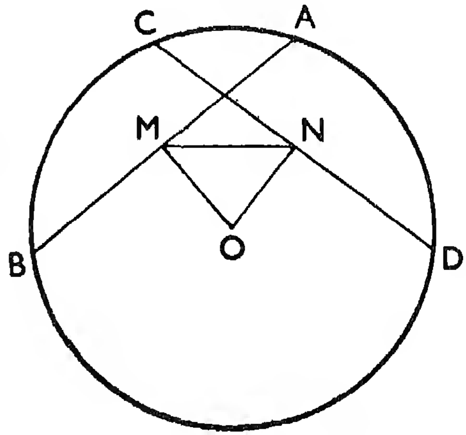


Fig. 22.26

11. In Fig. 22.27, AB and AC are equal chords of a circle with centre O. If $OM \perp AB$ and $ON \perp AC$, prove that

(*i*) $\triangle ABN \cong \triangle ACM$

(*ii*) $\angle MBN = \angle NCM$

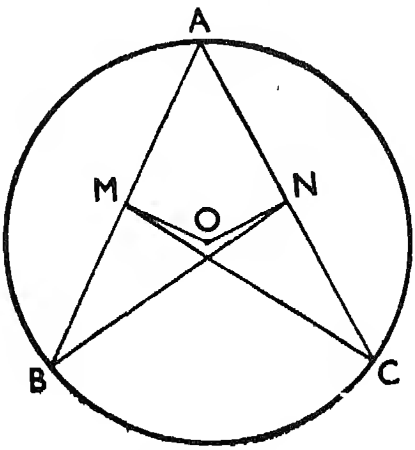


Fig. 22.27

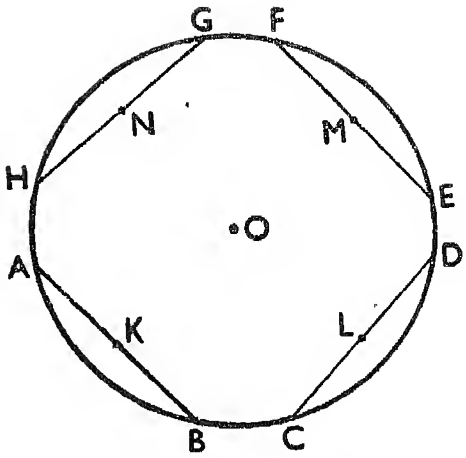


Fig. 22.28

12. In Fig 22.28, K, L, M and N are respectively the mid-points of the equal chords AB, CD, EF and GH of a circle with centre O.

Show that K, L, M and N lie on a circle with centre O

13. Determine the locus of the mid-points of the equal chords of a circle.

22.6. Arcs and Segments of a Circle

Let P and Q be any two points on a circle C. (See Fig. 22.29) These points divide the circle into two parts. Each part is called an **arc** of the circle.

To distinguish between the two arcs, we take an arbitrary point on each arc as shown in the figure. We, then, name the arcs as arc PAQ and arc PBQ.

When P and Q are the end-points of a diameter of the circle, each arc is called a **semi-circle.** In this case, the two arcs are, obviously, congruent.

An arc shorter than a semi-circle is called a **minor arc**. An arc longer than a semi-circle is called a **major arc.**

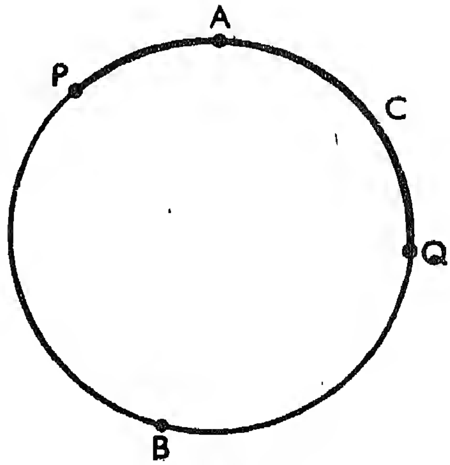


Fig. 22.29

Now, let AB be a chord of a circle with centre O (See Fig. 22 30)

The chord AB divides the circular region into two parts each of which is called a **segment of the circular region,** or briefly, a **segment of the circle.**

We note that **the major segment of the circle contains the centre of the circle.**

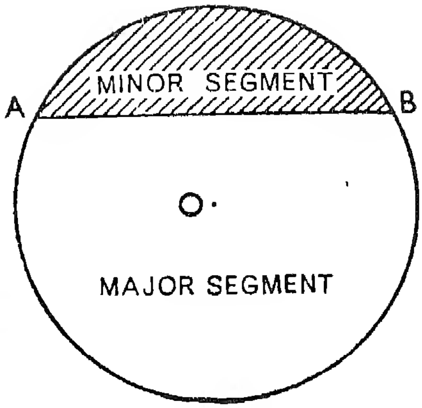


Fig. 22.30

Let OA and OB be two radii of a circle whose centre is O. (See Fig 22.31) The angle determined by these radii divides the circular region into two parts, each of which is called a **sector of the circular region,** or briefly, a **sector of the circle.**

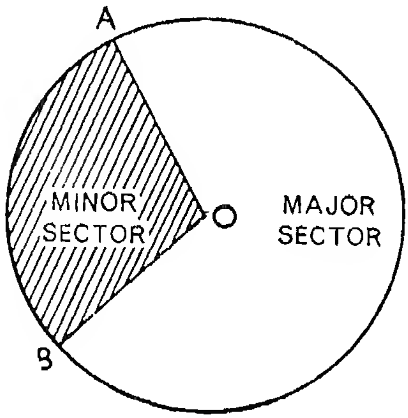


Fig. 22.31

Theorem 17 : **Angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.**

Given : A circle with centre O and an arc BMC. $\angle BOC$ is the angle subtended by the arc at the centre while $\angle BAC$ is the angle subtended at any point* A on the remaining part

*The reader may note that when BMC is a minor arc, two positions of A are of interest. These are shown in (a) and (c). The position of A when BMC is a major arc is shown in (b) (See Fig. 22 32)

of the circle. (See Fig. 22.32)

To prove : $\angle BOC = 2\angle BAC$

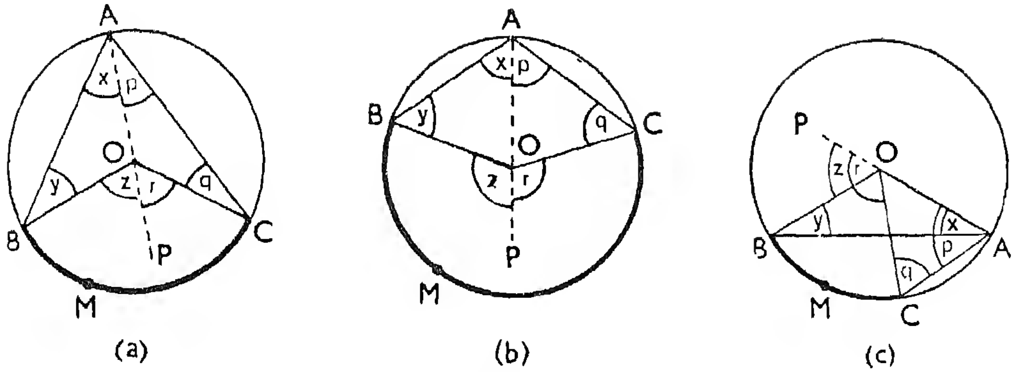


Fig. 22.32

Construction. Join A and O. Produce AO to some point P as shown in the figure.

Proof: In $\triangle OAB$,

$OA = OB$ (Why ?)

$\therefore$ $\angle x = \angle y$ (i)

Now, $\angle z = \angle x + \angle y$ (Corollary 1, Theorem 1)

$\therefore$ $\angle z = 2\angle x$ [From (i)] (ii)

Similarly, we can show that

$\angle r = 2\angle p$ (iii)

For figures 22·32 (*a*) and (*b*), we add (*ii*) and (*iii*) to obtain

$$\angle z + \angle r = 2(\angle x + \angle p)$$

i.e., $\angle BOC = 2\angle BAC$

For figure 22·32 (c), we subtract (*ii*) from (*iii*) to obtain

$$\angle r - \angle z = 2(\angle p - \angle x)$$

i.e., $\angle BOC = 2\angle BAC$

Q.E.D.

Corollary : **Angle in a semi-circle* is a right angle.**

Let BAC be a semi-circle of a circle whose centre is O. (See Fig. 22·33)

Then, $\angle BOC = 2\angle BAC$ (Theorem 17)

Whence, $\angle BAC = \frac{1}{2}\angle BOC = 90^\circ$

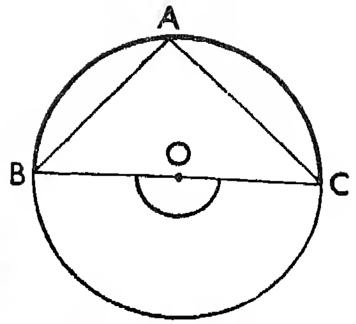


Fig. 22.33

Angle in a segment : By an angle in a segment of a circle, we mean, **the angle subtended by the chord of the segment at any point on the arc forming the segment.**

Theorem 18 : **Angles in the same segment of a circle are equal.**

Given : BAC and BDC are angles in the same segment formed by the chord BC of a circle whose centre is O. (See Fig. 22·34)

To Prove : $\angle BAC = \angle BDC$

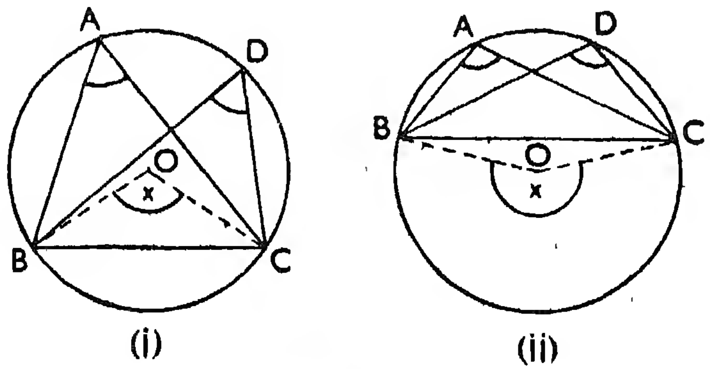


Fig. 22.34

Construction : Join B and O as also C and O.

Proof :	$\angle x = 2\angle BAC$	(Theorem 17)	(*i*)
Also,	$\angle x = 2\angle BDC$	(Theorem 17)	(*ii*)
$\therefore$	$2\angle BAC = 2\angle BDC$	[From (*i*) and (*ii*)]	
Whence,	$\angle BAC = \angle BDC$		

Q.E.D.

***By an angle in a semi-circle, we mean the angle subtended by the diameter (forming the semi-circle) at any point of the semi-circle.**

Converse of Theorem 18 : **If a line-segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on the same circle.**

Given : Line-segment BC subtending equal angles x and y at the points A and D lying on the same side of BC (See Fig. 22.35)

To Prove : Points A, B, C and D lie on the same circle.

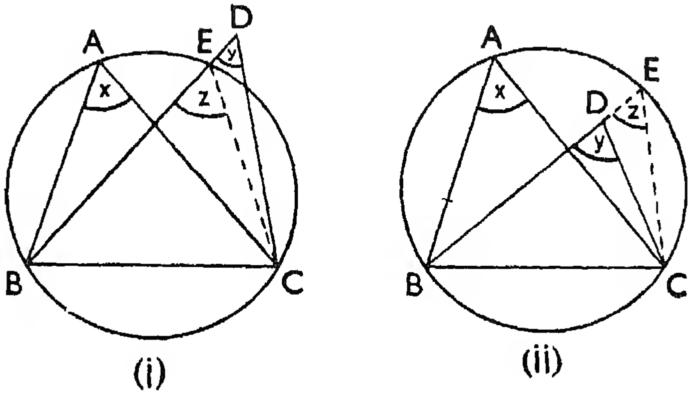


Fig. 22.35

Proof (by contradiction) : Let us assume that the four points do not lie on a circle. For convenience, let us say, that the circle through A, B and C does not pass through D and further that this circle meets BD or BD produced at the point E.

Now, $\angle x = \angle z$ (Theorem-18)

But, $\angle x = \angle y$ (Given)

$\therefore$ $\angle y = \angle z$

But this is impossible, since the exterior angle of a triangle is always greater than either of its interior opposite angles.

Hence, the circle passing through A, B and C must also pass through D.

In other words, the points A, B, C and D lie on the same circle.

Q.E.D.

The points lying on the same circle are called CONCYCLIC POINTS.

Example 1 : BC is a chord of a circle with centre O. A is a point on an arc as shown in Fig. 22.36. Prove that

(*a*) $\angle BAC + \angle OBC = 90°$, if A is a point on the major arc. [Fig. 22.36 (*i*)]

(*b*) $\angle BAC - \angle OBC = 90°$, if A is a point on the minor arc. [Fig. 22.36 (*ii*)]

Solution : (*a*) $\angle z = 2\angle x$ (Why ?)

Also, $\angle z = 180° - 2\angle y$

$\therefore$ $2\angle x + 2\angle y = 180°$

Whence, $\angle BAC + \angle OBC = 90°$

(*b*) Again, $\angle z = 2\angle x$ (i)

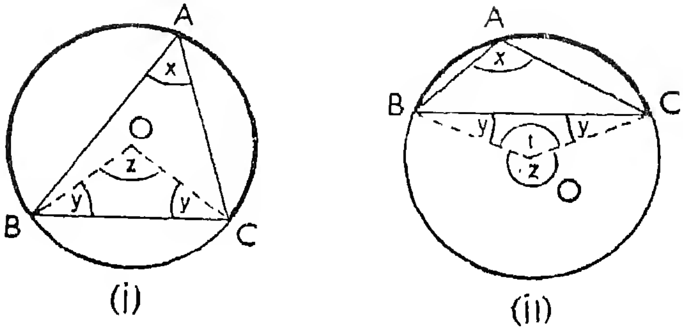


Fig. 22.36

And, $\angle t = 180° - 2\angle y$

Also, $\angle t = 360° - \angle z$

$\therefore$ $360° - \angle z = 180° - 2\angle y$

or, $360° - 2\angle x = 180° - 2\angle y$ [From (i)]

Whence, $\angle x - \angle y = 90°$

i.e., $\angle BAC - \angle OBC = 90°$

Example 2 : In Fig. 22.37, PQ and RQ are chords of a circle, equidistant from the centre O. Prove that the diameter QS bisects the angles PQR and PSR.

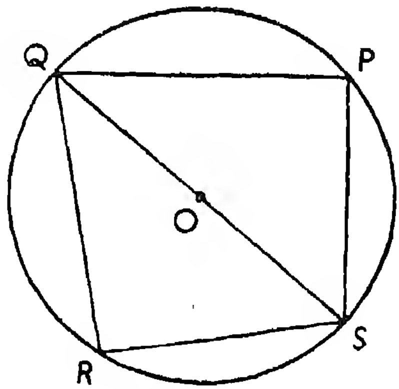


Fig. 22.37

Solution : Since PQ and RQ are equidistant from O

$PQ = RQ$ (Why ?)

Now, in triangles PQS and RQS,

$$\angle QPS = \angle QRS = 90^\circ \qquad \text{(Angles in a semi-circle)}$$

$\therefore$ $\triangle PQS \cong \triangle RQS$ (RHS)

Whence, $\angle PQS = \angle RQS$

and, $\angle PSQ = \angle RSQ$

i.e., QS bisects $\angle PQR$ and $\angle PSR$

Exercise 22.3

1. In Fig. 22.38, A, B and C are three points on a circle such that the angles subtended by the chords AB and AC at the centre O are of 90° and 110° respectively. Determine $\angle BAC$.

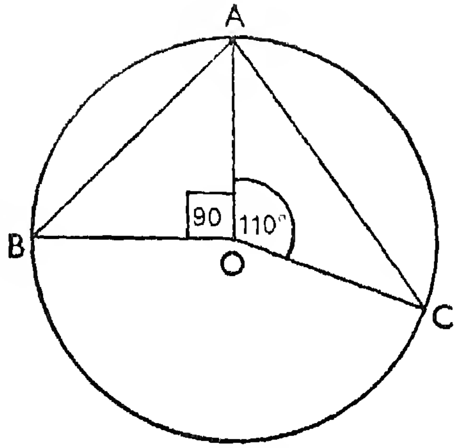


Fig. 22.38

2. In Fig. 22.39, a diameter AB of a circle bisects a chord PQ. If $AQ \parallel PB$, prove that the chord PQ is also a diameter of the circle.

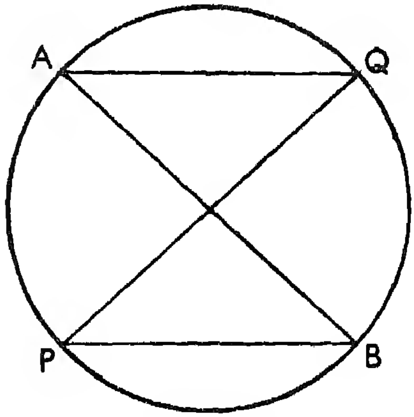


Fig. 22.39

3. In Fig 22.40, AC and BD are chords that bisect each other Prove that

   (*i*) AC and BD are diameters.

   (*ii*) $ABCD$ is a rectangle.

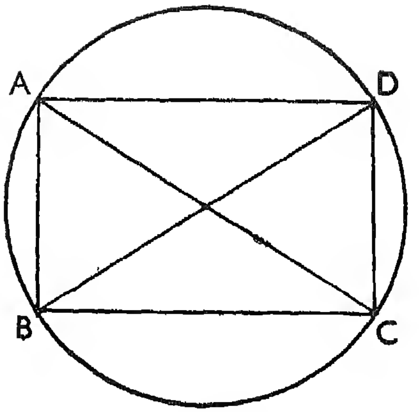


Fig. 22.40

4. Two diameters of a circle intersect each other at right angles. Prove that the quadrilateral formed by joining their end-points is a square.
5. Prove that the angle in a major segment is acute and that in a minor segment is obtuse.
6. In Fig. 22.41, D is the mid-point of the side BC of a triangle ABC. Also, $AB = AC$. Prove that the circle drawn with either of the equal sides as a diameter passes through the point D.

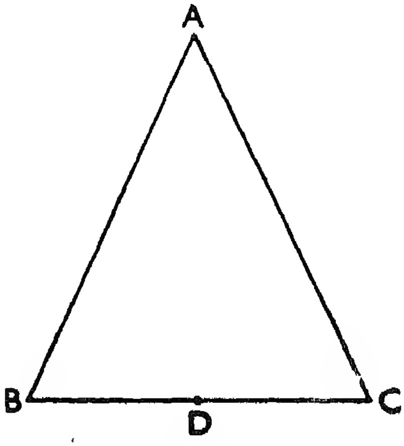


Fig. 22.41

7. "The circle drawn with the hypotenuse of a right triangle as a diameter passes through the opposite vertex". Prove.
   (This result is the converse of the corollary to Theorem 17.)
8. Prove that the circle, drawn with any side of a rhombus as a diameter, passes through the point of intersection of its diagonals.

9. In Fig. 22.42, $ABCD$ is a quadrilateral whose vertices lie on a circle. Also, $AD \parallel BC$. Prove that $AB = DC$.

(**Hint :** Join A and C as also B and D.)

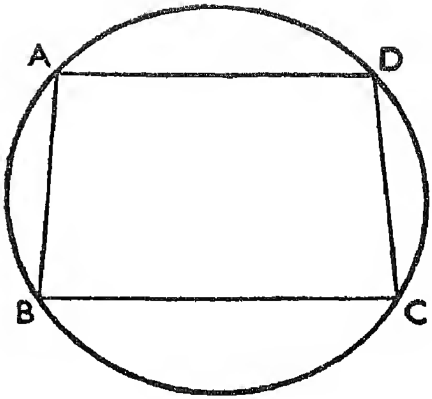


Fig. 22.42

10. In Fig. 22.43, $ABCD$ is a quadrilateral whose vertices lie on a circle. The diagonals of the quadrilateral intersect at O. Also $AB = DC$.

Prove that :

(*i*) $\triangle OAB \cong \triangle ODC$

(*ii*) $OA = OD$ and $OC = OB$

(*iii*) $AD \parallel BC$

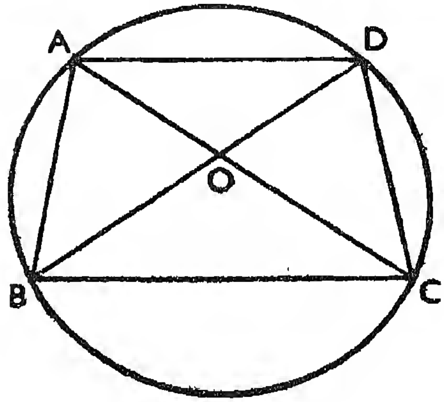


Fig. 22.43

11. In Fig. 22.44, $AB = CD$. Prove that $BE = DE$ and $AE = CE$.

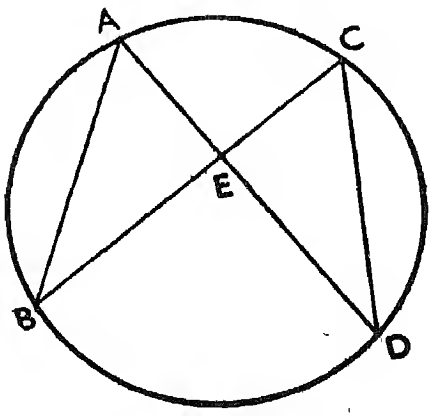


Fig. 22.44

12. In Fig. 22.45, ABC and ADC are right triangles with the common hypotenuse AC. Prove that

$$\angle CAD = \angle CBD$$

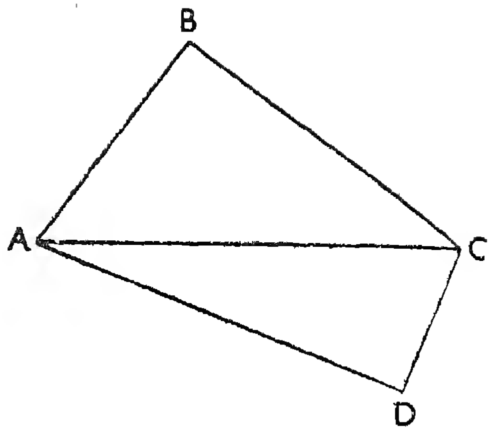


Fig. 22.45

*13 Show that the circles drawn with any two sides of a triangle as their respective diameters intersect on the line containing the third side

*14 In Fig. 22.46, the diagonals AC and BD of a quadrilateral $ABCD$, whose vertices lie on a circle, intersect at right angles at the point M.

Prove that a line drawn through M to bisect any side of the quadrilateral is perpendicular to the opposite side.

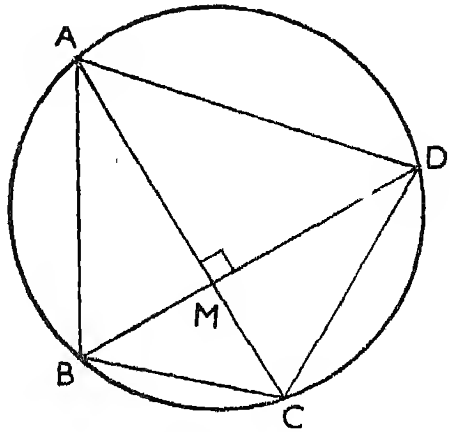


Fig. 22.46

15. In Fig. 22.47, ABC is a triangle and P is a point on the side BC such that $AB = AP$. If AP produced intersects the circumcircle of $\triangle ABC$ at the point Q, prove that $CP = CQ$.

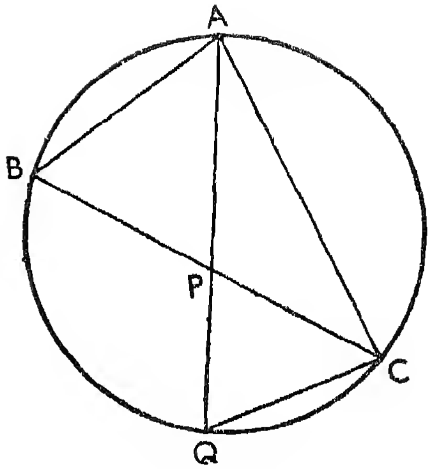


Fig. 22.47

16 Two circles intersect each other at the points A and B. If AC and AD are respectively the diameters of the two circles, show that the points B, C and D are collinear.

17 D is a point on the circumcircle of a triangle ABC in which $AB = AC$ such that B and D are on the opposite sides of line AC. If CD is produced to a point E such that $CE = BD$, prove that $AD = AE$.

22.7. Angles of a Cyclic Quadrilateral

Cyclic quadrilateral : **A quadrilateral whose vertices lie on a circle is called a CYCLIC QUADRILATERAL.**

Theorem 19 : **The sum of the opposite angles of a cyclic quadrilateral is 180°, i.e., they are supplementary.**

Given : A cyclic quadrilateral $ABCD$. O is the centre of the circle containing the vertices of the quadrilateral. (See Fig. 22.48)

To prove : $\angle BAD + \angle BCD = 180°$

and $\angle ABC + \angle ADC = 180°$

Proof : $\angle BAD = \frac{1}{2}\angle x$ (Theorem 17)

And, $\angle BCD = \frac{1}{2}\angle y$ (Theorem 17)

$\therefore$ $\angle BAD + \angle BCD = \frac{1}{2}(\angle x + \angle y) = 180°$

It follows that

$$\angle ABC + \angle ADC = 180^\circ$$

Q.E.D.

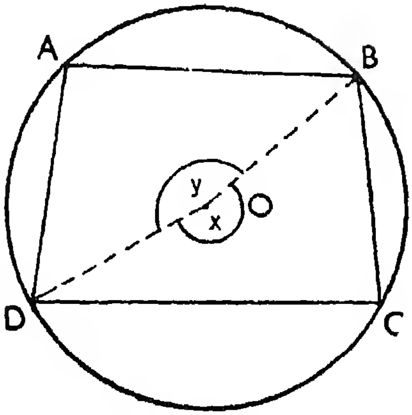


Fig. 22.48

Converse of Theorem 19 : **If a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.**

Given : A quadrilateral $ABCD$ in which

$$\angle ABC + \angle ADC = 180^\circ. \quad \text{(See Fig. 22.49)}$$

To prove : $ABCD$ is a cyclic quadrilateral.

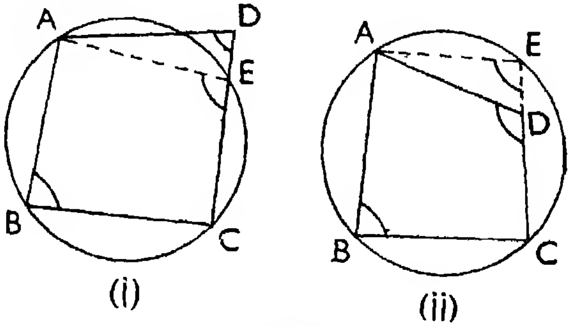


Fig. 22.49

Proof (by contradiction) : Let us assume that the quadrilateral is not cyclic. For convenience, let us say that the circle passing through A, B and C does not pass through D.

Further, let this circle meet the side CD or CD produced at E.

Then, $ABCE$ is a cyclic quadrilateral.

Whence, $\angle ABC + \angle AEC = 180°$ (Theorem 19)

But, $\angle ABC + \angle ADC = 180°$ (Given)

$\therefore$ $\angle AEC = \angle ADC$

But, this is impossible. (Why ?)

Hence, the circle passing through A, B and C must also pass through D.

Thus, $ABCD$ is a cyclic quadrilateral.

Q.E.D.

Example 1 : Given a cyclic trapezium $ABCD$ in which $AD \parallel BC$. Also, $\angle B = 70°$. (See Fig. 22.50) Determine its other three angles.

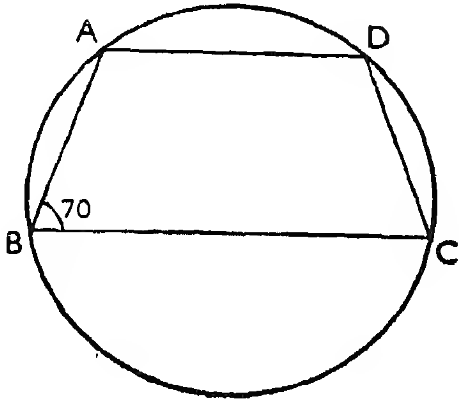


Fig. 22.50

Solution : $AD \parallel BC$

$\therefore$ $\angle A + \angle B = 180°$

Whence, $\angle A = 110°$ (Since, $\angle B = 70°$)

Also, $\angle A$ and $\angle C$ are opposite angles of a cyclic quadrilateral.

$\therefore$ $\angle A + \angle C = 180°$ (Theorem 19)

Thus, $\angle C = 70°$

It follows that, $\angle D = 110°$

The other three angles of the trapezium are, therefore, 110°, 110° and 70°.

Example 2 : $ABCD$ is a cyclic quadrilateral. A circle passing through A and B meets AD and BC in the points E and F respectively. (See Fig. 22.51) Prove that $EF \parallel DC$.

Solution : Since $ABCD$ is a cyclic quadrilateral

$\therefore$ $\angle x + \angle y = 180°$ (*i*)

Also, $ABFE$ is a cyclic quadrilateral

$\therefore$ $\angle x + \angle z = 180°$ (*ii*)

Whence, $\angle y = \angle z$ [From (*i*) and (*ii*)]

Thus, $EF \parallel DC$. (Corresponding angles are equal)

(The reader is advised to prove the above result when the circle through A and B intersects AD and BC produced at the points E and F respectively.)

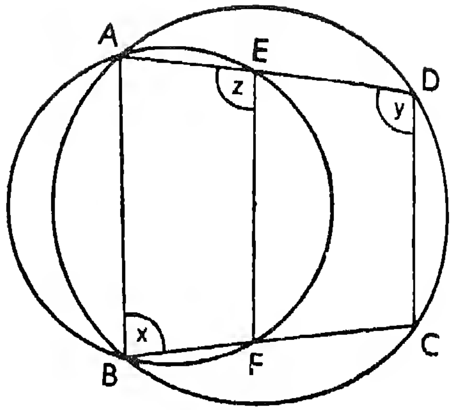


Fig. 22.51

Exercise 22.4

1. Prove that the exterior angle, formed by producing a side of a cyclic quadrilateral, is equal to the interior opposite angle.

2. Opposite sides AD and BC of a cyclic quadrilateral are produced to meet at a point P. Prove that $\triangle PCD \sim \triangle PAB$.

3. In Fig. 22.52, P is a point on the side BC of a $\triangle ABC$ such that $AB = AP$. From A and C, lines are drawn parallel to BC and PA respectively so as to intersect at the point D. Show that $ABCD$ is a cyclic quadrilateral.

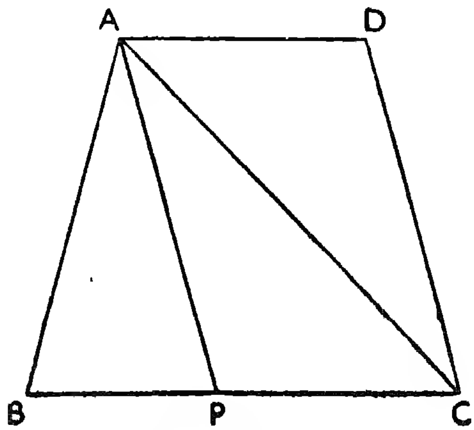


Fig. 22.52

4. In Fig. 22.53, ABC is a triangle in which $AB=AC$. Also, a circle passing through B and C intersects the sides AB and AC at the points D and E respectively. Prove that $DE \parallel BC$.

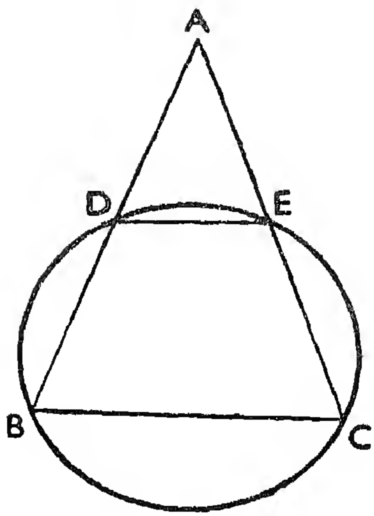


Fig. 22.53

5. Show that any cyclic parallelogram is a rectangle.

6. Prove that the quadrilateral formed, if possible, by the bisectors of the angles of a quadrilateral is cyclic.

*7. The bisectors of the opposite angles A and C of a cyclic quadrilateral intersect the circle at the points E and F respectively as shown in Fig. 22.54. Prove that EF is a diameter of the circle.
(**Hint :** Join A and F)

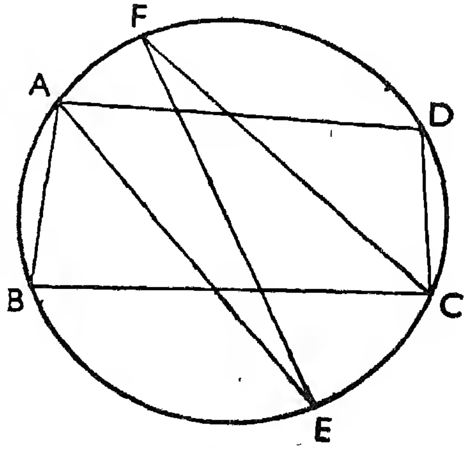


Fig. 22.54

*8. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect each other at right angles at the point E as shown in Fig. 22.55. The line through E, perpendicular to AB, meets CD at F. Show that F is the mid-point of CD.

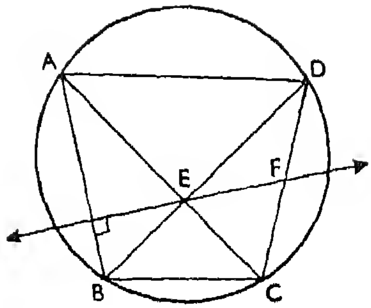


Fig. 22.55

9. The circle passing through the vertices A, B and C of a parallelogram $ABCD$ intersects the side CD (or CD produced) at the point P.
Prove that $AP = AD$.

10. In Fig. 22.56, PQ and RS are parallel chords of a circle and the lines RP and SQ intersect each other at the point O.

Prove that $OP = OQ$.

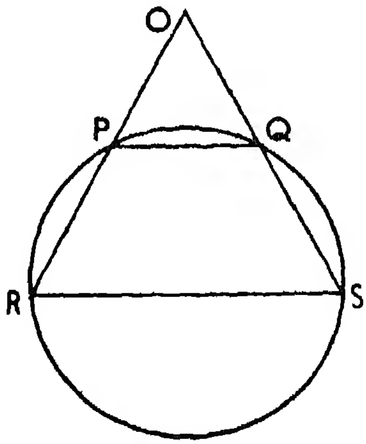


Fig. 22.56

*11. The bisector of $\angle ABC$ of a triangle ABC, in which $AB = AC$, intersects the circumcircle of the triangle at P. Line AP intersects BC produced at Q. (See Fig. 22.57)

Show that $CQ = CA$.

(**Hint** : Join C and P)

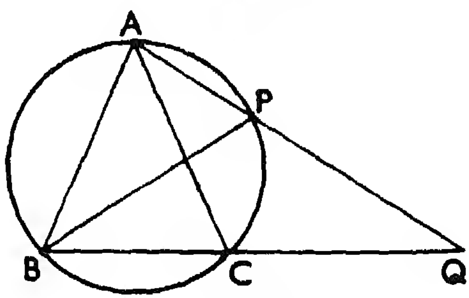


Fig. 22.57

12. In Fig. 22.58, ABC is a triangle in which $AB = AC$ and P is a point on AC. Through C, a line is drawn to intersect BP produced at Q such that $\angle ABP = \angle ACQ$.

Prove that $\angle AQC = 90° + \frac{1}{2}\angle BAC$.

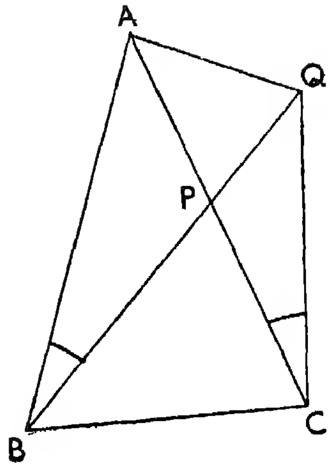


Fig. 22.58

13. D and E are respectively the points on equal sides AB and AC of a triangle ABC such that $AD = AE$.
Prove that points B, C, E and D are concyclic.

*14. In Fig. 22.59, D and E are respectively the points on equal sides AB and AC of a triangle ABC such that points B, C, E and D are concyclic. If O is the point of intersection of CD and BE, prove that AO is the perpendicular bisector of the segment DE.

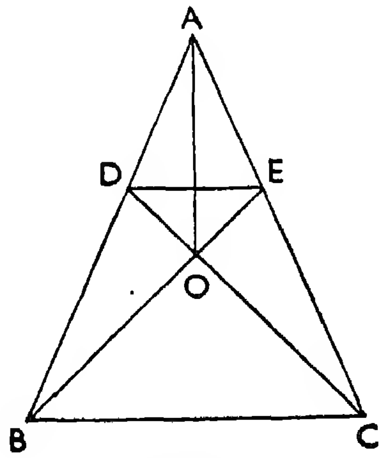


Fig. 22.59

*15. Prove that the altitudes of a triangle are concurrent.

(*Note* : **The point of intersection of the altitudes of a triangle is called the ORTHO-CENTRE of the triangle.**)

22·8. Congruent Arcs

Let us consider two congruent circles with centres O and O'. Further, let us draw two radii OA and OB of the circle whose centre is O.

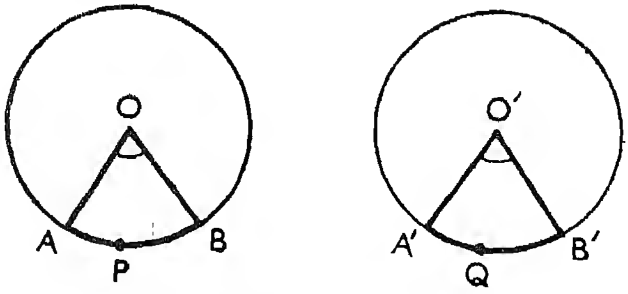


Fig. 22.60

Through the centre O' of the other circle, let us draw radii $O'A'$ and $O'B'$ such that

$$\angle A'O'B' = \angle AOB \quad \text{(See Fig. 22.60)}$$

Now, let us cut off the sector $A'QB'O'$ and place it on the sector $APBO$ in such a way that O' coincides with O and $O'A'$ falls along OA.

Since $O'A' = OA$, A' will fall on A.

Also, since $\angle A'O'B' = \angle AOB$, $O'B'$ will fall along OB.

Further since $O'B' = OB$, B' will fall on B.

It will, therefore, be seen that the arc $A'QB'$ completely covers the arc APB.

In other words

In two congruent circles (or in the same circle), if the angles subtended by two arcs at the centres of their corresponding circles are equal, then the two arcs are congruent.

In a similar way, we can observe that if the angles subtended at the centre are not equal, then the corresponding arcs are not congruent.

It follows, therefore, that **congruent arcs of two congruent circles (or in the same circle) subtend equal angles at their centres.**

As a simple consequence of Theorem 17, we can further conclude that

In two congruent circles (or in the same circle), if the angles subtended by two arcs at points on the remaining parts of their corresponding circles are equal, then the two arcs are congruent and conversely.

Let us now consider Fig. 22.61 in which AB and $A'B'$ are equal chords of two congruent circles whose centres are O and O' respectively.

It is obvious that $\triangle OAB \cong \triangle O'A'B'$ (SSS)

$\therefore$ $\angle AOB = \angle A'O'B'$

Whence, arc $APB \cong$ arc $A'QB'$

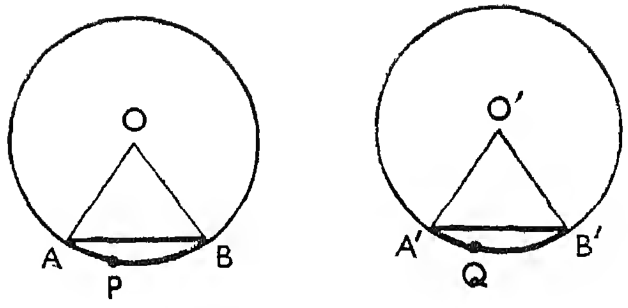


Fig. 22.61

In other words

If two chords of congruent circles (or in the same circle) are equal, the corresponding arcs are congruent.

Conversely, if arc $APB \cong$ arc $A'QB'$, then

$$\angle AOB = \angle A'O'B' \qquad \text{(Why ?)}$$

$\therefore$
$$\triangle AOB \cong \triangle A'O'B' \qquad (SAS)$$

Whence,
$$AB = A'B'$$

Thus, we see that

If two arcs of congruent circles (or in the same circle) are congruent, the corresponding chords are equal.

Example 1 : Through a point of intersection A of the two congruent circles, segments MAN and RAS are drawn as shown in Fig. 22.62. Prove that the chords MR and NS are equal.

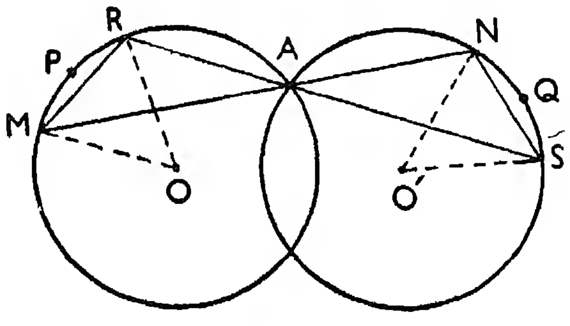


Fig. 22.62

Solution : Let O and O' be the respective centres of the two circles.

$\angle RAM = \angle NAS$ (Vertically opposite angles) (*i*)

But, $\angle ROM = 2\angle RAM$ (Why ?)

And, $\angle NO'S = 2\angle NAS$

$\therefore$ $\angle ROM = \angle NO'S$

$\therefore$ arc $MPR \cong$ arc NQS (*ii*)

Whence, $MR = NS$

[The reader is advised to note that, in congruent circles, if the angles subtended by two arcs at points on the remaining parts of their corresponding circles are equal, the two arcs are congruent We could have, therefore, used this result to conclude (*ii*) directly from (*i*). We could have thus done away with the intermediate steps.]

Example 2 : Prove that the two arcs intercepted between two parallel chords of a circle are congruent.

Solution : Let AB and CD be two parallel chords of a circle. (See Fig. 22.63) We need to show that arc $APC \cong$ arc BQD

Let us join A and D.

We observe that $\angle BAD$ and $\angle CDA$ are alternate angles.

But $AB \parallel CD$ (Given)

$\therefore$ $\angle CDA = \angle BAD$

Whence, arc $APC \cong$ arc BQD

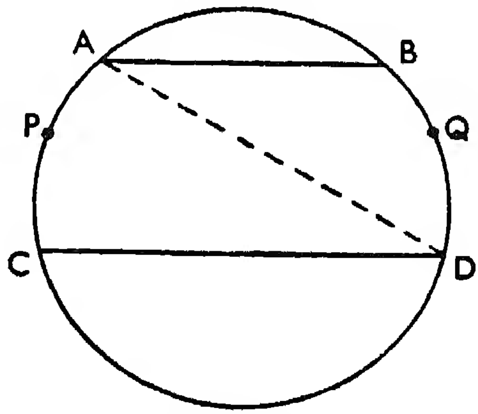


Fig. 22.63

Example 3 : Perpendiculars from vertices B and C of a triangle ABC to their opposite sides intersect the circumcircle of the triangle at the points P and Q respectively as shown in Fig. 22.64. Show that arc $ARP \cong$ arc ASQ.

Solution : Let the segments BP and CQ intersect the sides CA and AB at the points M and N respectively.

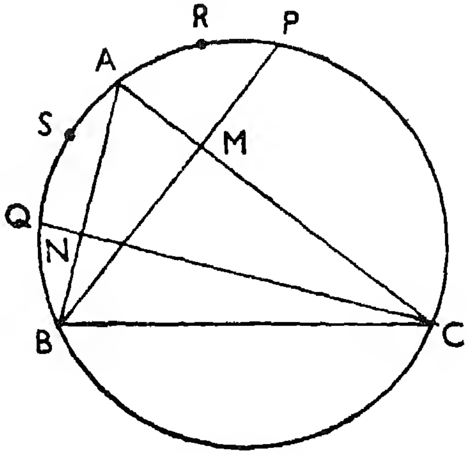


Fig. 22.64

Now in triangles ABM and ACN

$\angle AMB = \angle ANC$ (Each of 90°)

And, $\angle A = \angle A$

$\therefore$ $\angle ABM = \angle ACN$

or, $\angle ABP = \angle ACQ$

Hence, arc $ARP \cong$ arc ASQ

Exercise 22.5

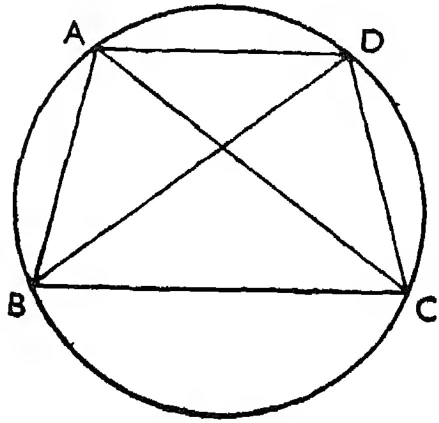


Fig. 22.65

1. In Fig. 22.65, the diagonals AC and BD of a cyclic quadrilateral $ABCD$ are equal and $AD \neq BC$

   Prove that $AB = CD$.

2. A pair of opposite sides of a cyclic quadrilateral are equal. Prove that its diagonals are equal.

3. In Fig 22·66, $ABCD$ is a cyclic square and E is a point on the arc APB. (E is not the point A or B.) Prove that ED and EC trisect $\angle AEB$.

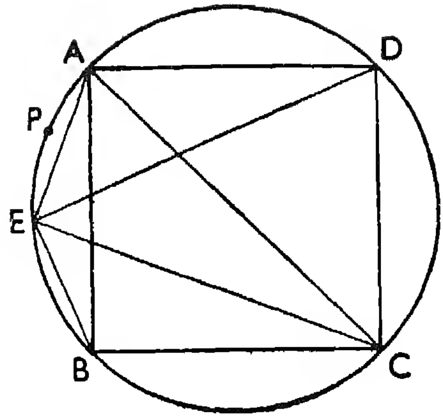


Fig. 22.66

4. AB and CD are chords of a circle. The chords intersect at the point P as shown in Fig. 22.67.

   (a) If arc $ARC \cong$ arc BSD, prove that $CP = BP$

   (**Hint :** Join A and C as also B and D.)

   (b) If $CP = BP$, prove that arc $ARC \cong$ arc BSD

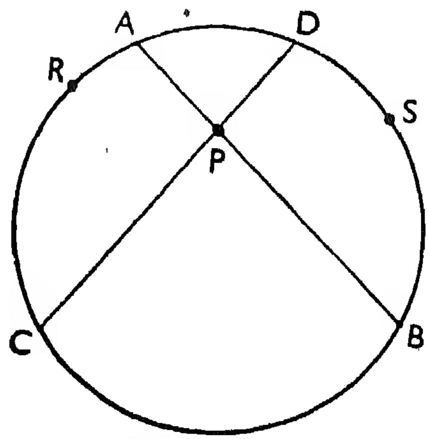


Fig. 22.67

5 In Fig 22.68, A, B, C and D are four concyclic points such that arc $APB \cong$ arc DQC. Prove that $\angle ABC = \angle DCB$

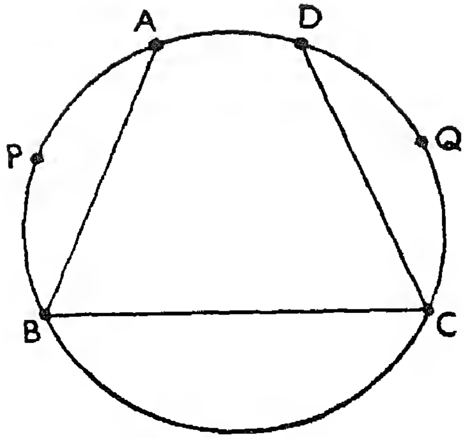


Fig. 22.68

6. In Fig. 22.69, the chord PQ of a circle, whose centre is O, has been produced on both the sides to points R and S respectively such that $PR = QS$. If OR and OS intersect the circle at the points T and U respectively, prove that

(i) $\angle OPR = \angle OQS$

(ii) arc $PLT \cong$ arc QMU

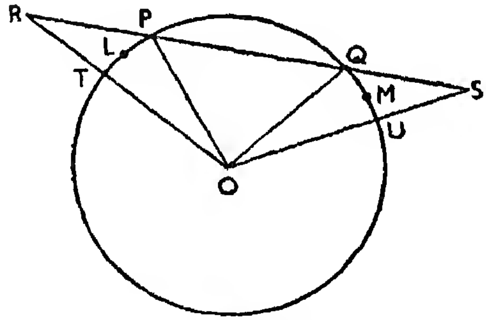


Fig. 22.69

7. Two chords AB and CD of a circle intersect (when produced) at E such that $EB = ED$ as shown in Fig. 22.70.

Prove that

(i) arc $DBA \cong$ arc BDC

(ii) $AB = CD$

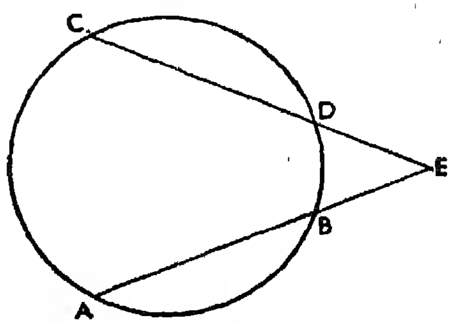


Fig. 22.70

8. In Fig 22.71, AB and AC are chords of a circle. l is a line that intersects the circle at P and Q and the chords AB and AC at R and S respectively in such a way that minor arc $AP \cong$ minor arc AQ and minor arc $PB \cong$ minor arc QC.

   Prove that $AR = AS$.

   (**Hint :** Join P and C as also Q and B.)

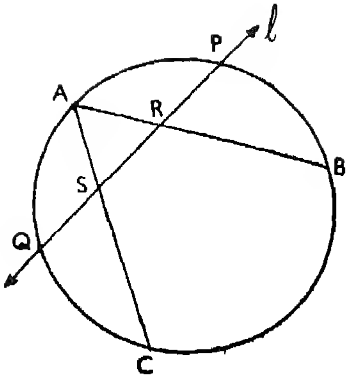


Fig. 22.71

9. Two chords AB and CD of a circle, with centre O, intersect when produced at P as shown in Fig. 22.72. Join the points B and C, and show that
   $\angle APC = \frac{1}{2}(\angle AOC - \angle BOD)$

   (**Note** : This result may be alternatively stated as :

If two chords of a circle intersect (when produced) at a point outside the circle, the angle between them is equal to half the difference of the angles subtended at the centre by the two opposite arcs cut off in between the two chords.)

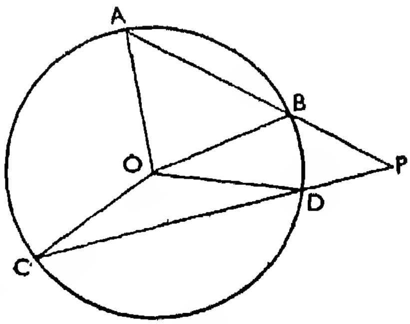


Fig. 22.72

10. Two chords of a circle intersect at a point inside the circle. Prove that the angle between the two chords is equal to half the sum of the angles subtended at the centre by the two opposite arcs cut off in between the two chords.

11. In Fig. 22.73, M divides the arc APB of a circle, with centre O, into two congruent parts.

Prove that $\angle BAM = \frac{1}{2}\angle BOA$.

(**Hint :** Join O and M)

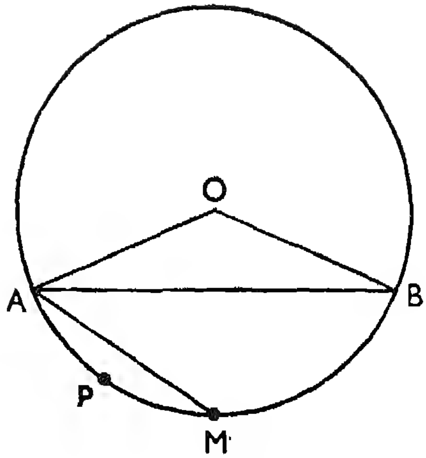


Fig. 22.73

12. In Fig. 22.74, l is the perpendicular bisector of the side BC of a $\triangle ABC$ meeting the circumcircle of the triangle at the point M.

Show that

(*i*) Centre of the circle lies on l.

(*ii*) M bisects arc BPC.

(*iii*) AM is the bisector of $\angle BAC$.

(**Note :** The above may be alternatively stated as :

The bisector of any angle of a triangle and the perpendicular bisector of its opposite side intersect on the circumcircle of the triangle.)

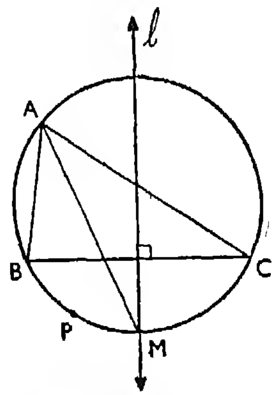


Fig. 22.74

13. In Fig. 22.75, two circles intersect at the points A and B. Two lines, through A, meet the circles in the points D, G and E, F as shown Show that $\angle DBG = \angle EBF$.

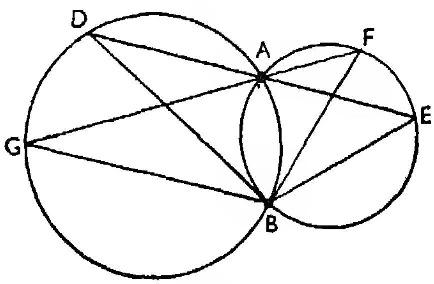


Fig. 22.75

14 In Fig. 22.76, AB and AC are chords of a circle M and N are the points on the circle, such that M bisects the arc ARB and N bisects the arc ASC If MN intersects the chords AB and AC at the points P and Q respectively, prove that

(i) $\angle PMA = \angle QAN$

(ii) $\angle PAM = \angle QNA$

(iii) $AP = AQ$

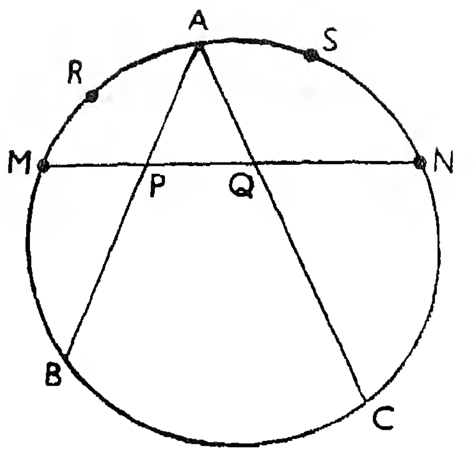


Fig. 22.76

15. In Fig. 22.77, ARB and CSD are two congruent arcs. Show that $AC = BD$

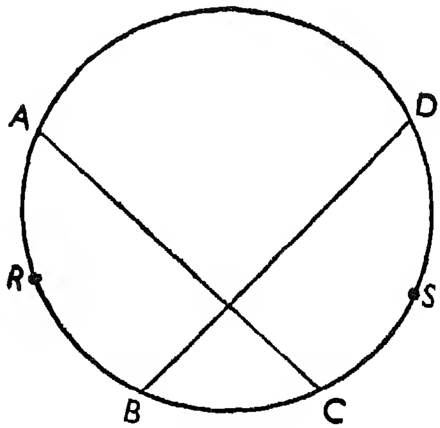


Fig. 22.77

16 In Fig 22.78, ARB and CSD are two congruent arcs. Show that

$AC \parallel BD$

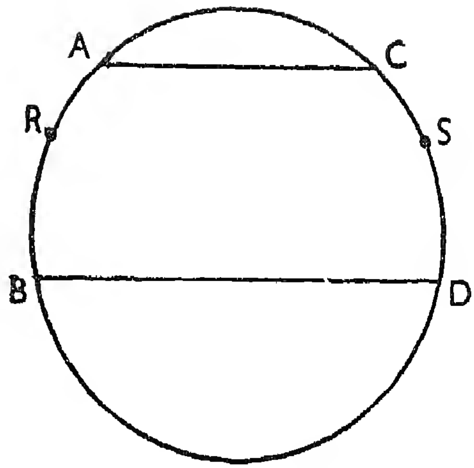


Fig. 22.78

(The two questions 15 and 16 above can be stated as :

The segments joining the end-points of two congruent arcs of a circle are equal or parallel.)

17. Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at the points D, E and F respectively. Prove that the angles of $\triangle DEF$ are $90° - \frac{A}{2}$, $90° - \frac{B}{2}$ and $90° - \frac{C}{2}$.

18. Take five points A, B, C, D and E such that no three are collinear. If A, B, C and D and B, C, D and E are concyclic, show that the five points are concyclic.

CHAPTER XXIII

SECANTS AND TANGENTS

23.1. Intersection of a Circle and a Line

In Fig. 23.1, l, m and n are three lines in the plane of the circle C. We note that l intersects the circle in two **distinct points**, m touches the circle at one point (i.e , intersects the circle in two **coincident** points) and n does not intersect the circle at all. In other words, l has two points in common with C, m has just one point in common with C and n has no point in common with C.

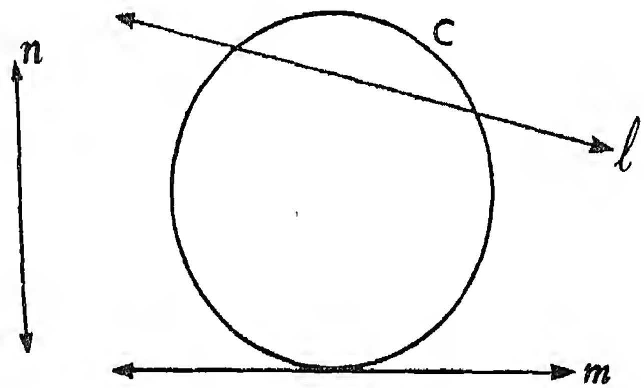


Fig. 23.1

We may also note that **no line can meet the circle in more than two points.** For, if a line were to meet a circle, say, in three points, it will follow that there is a circle through three collinear points which contradicts our earlier result that no circle can be made to pass through three collinear points.

A line which intersects a circle in two points is called a SECANT to the circle. In Fig. 23.2, PQ is a secant to the circle C.

Let us rotate the secant PQ about the point P until Q coincides with P and the secant attains the position l. At this position, the secant is called a **tangent** to the circle at the point P and P is called the **point of contact.**

In other words

A TANGENT is a secant which intersects the circle in two coincident points.

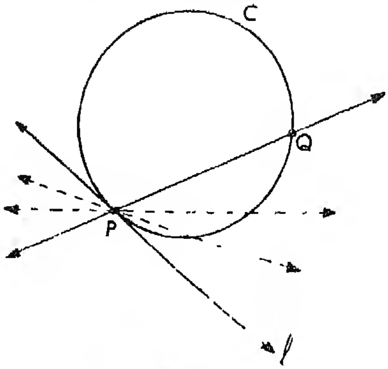


Fig. 23.2

Note 1

It is obvious that every point of the tangent, other than the point of contact, lies outside the circle.

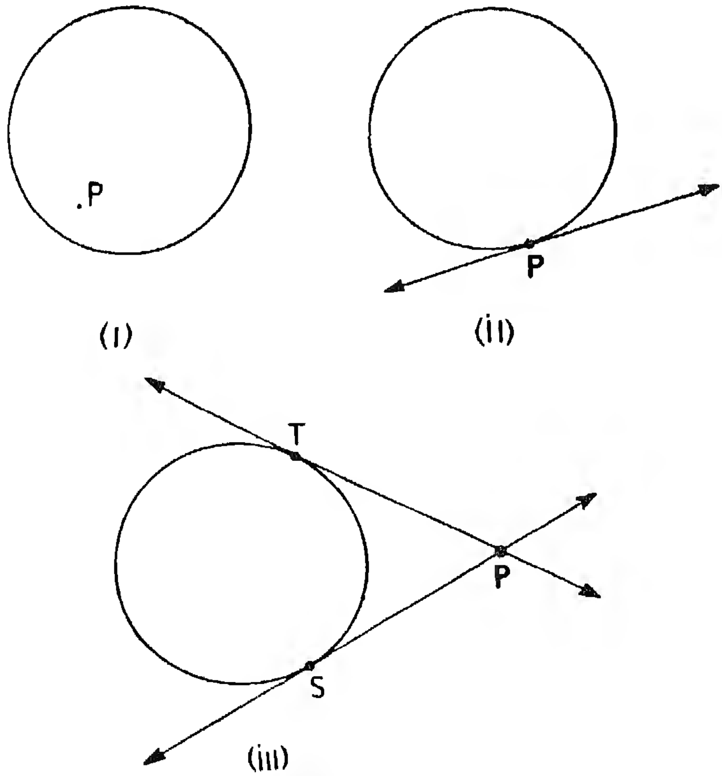


Fig. 23.3

Note 2 : (Fig. 23·3)

(*i*) If a point P lies inside a circle, no line passing through P can be a tangent to the circle. (See Theorem 20)

(*ii*) If P lies on the circle, then exactly one tangent can be drawn to pass through P.

(*iii*) If P lies outside the circle, then exactly two tangents can be drawn through P.

In Fig. 23.3 (*iii*), T and S are the points of contact of the two tangents through P. PT and PS are called the **lengths of the tangents** drawn from the external point P to the circle.

23.2. Circles and Tangents

Theorem 20 : **A tangent at any point of a circle is perpendicular to the radius through the point of contact.**

Given : A circle with centre O and l, a tangent to the circle at the point P. (See Fig 23.4)

To prove : $OP \perp l$

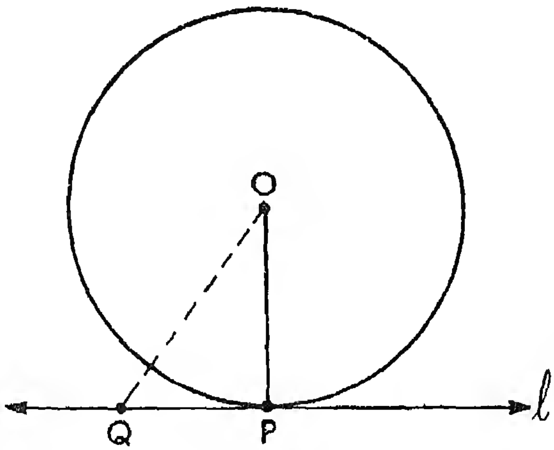


Fig. 23.4

Proof : Let Q be any point, other than P, on l. Since Q lies outside the circle,

$$OQ > OP$$

It follows that of all the segments that can be drawn to the line l from the point O, OP is the shortest.

$$\therefore \quad OP \perp l$$

Q.E.D.

(The reader is advised to refer to Section 13 6, Part I of this book.)

Theorem 21 : **The lengths of the two tangents to a circle from an external point are equal.**

Given : A circle with centre O. PT and PS are two tangents from an external point P. (See Fig 23.5)

To prove : $PS = PT$

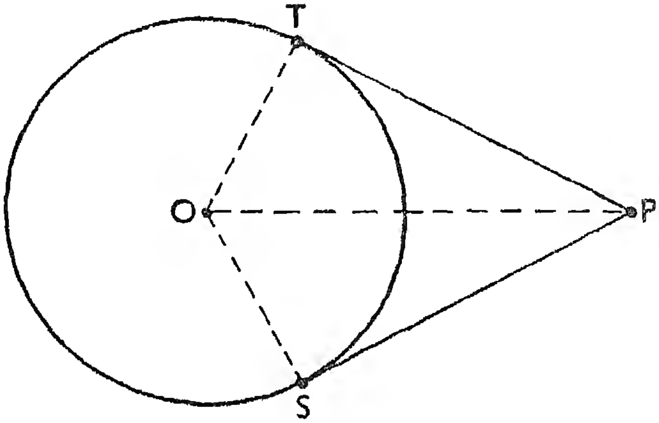


Fig. 23.5

Construction : Join P, T and S to the centre O.

Proof : Since a tangent is perpendicular to the radius through the point of contact,

$$\angle OSP = \angle OTP = 90°$$

Now, in right triangles PSO and PTO

$$OS = OT$$

$\therefore$ $\triangle PSO \cong \triangle PTO$ (RHS)

Whence, $PS = PT$

Q.E.D.

Example 1 : Given a point P on a circle, draw the tangent to the circle at the point P.

Solution : Let O be the centre of the circle. (See Fig. 23.6)

We go through the following steps to draw the required tangent :

Step 1 : Join O and P.

Step 2 : Through P, draw a line $l \perp OP$.

Then, l is the required tangent to the circle at the given point P.

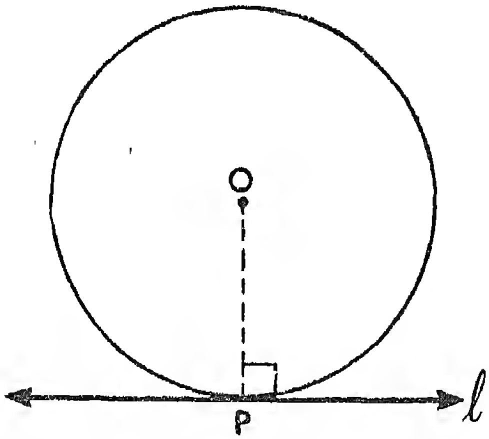


Fig. 23.6

Example 2 : Given a point P outside a circle, draw two tangents to the circle from the point P.

Solution : Let O be the centre of the circle. (See Fig. 23.7)

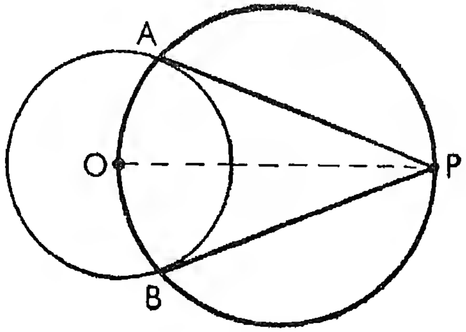


Fig. 23.7

The following steps are necessary for drawing the required tangents :

Step 1 : Join O and P.

Step 2 : With OP as diameter, draw a circle to intersect the given circle at the points A and B.

Step 3 : Join P and A as also P and B.

PA and PE are the required tangents.

Example 3 : Given a $\triangle ABC$, draw a circle touching the three sides of the triangle.

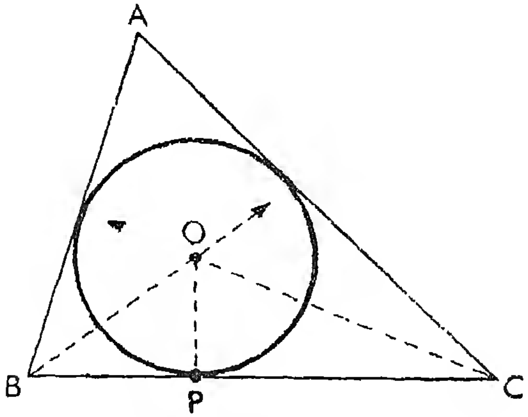


Fig. 23.8

Solution : We go through the following steps :

Step 1 : Draw the bisectors of $\angle B$ and $\angle C$. Denote their point of intersection by O

Step 2 : Draw $OP \perp BC$.

Step 3 : With O as centre and OP as radius, draw a circle.
This is the required circle.

[The circle which touches all the sides of a triangle is called the **incircle** of the triangle and the centre of the circle is called the **incentre** of the triangle.]

Example 4 : Given a triangle ABC, draw a circle touching one of its sides and the other two sides produced.

Solution : We shall draw the circle touching the side BC and the sides AB and AC produced.

The following steps are needed:

Step 1 : Produce AB to D and AC to E.

Step 2 : Draw the bisectors of $\angle DBC$ and $\angle ECB$. Denote their point of intersection by O.

Step 3 : Draw OP perpendicular to AC produced.

Step 4 : With centre O and OP as radius, draw a circle.
This is the required circle. (See Fig. 23.9)

[The circle is called an **excircle** of the triangle. The reader should note that there are **three** possible excircles of a triangle.]

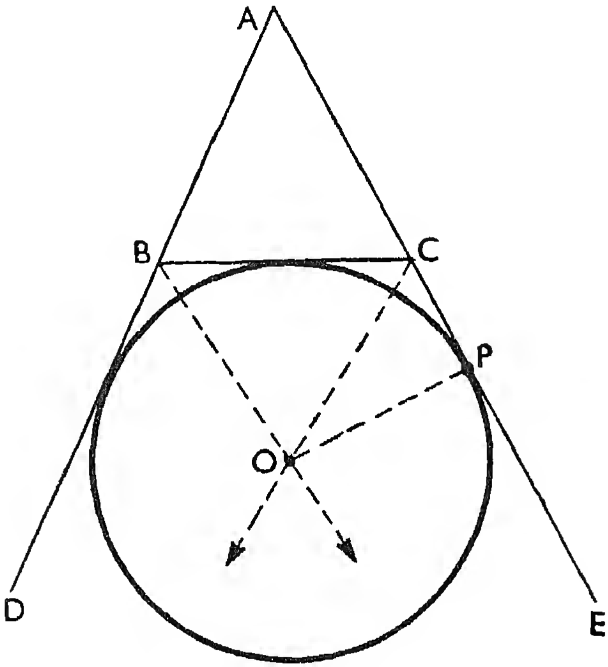


Fig. 23.9

23.3. Common Tangents to Two Circles

A tangent which touches two given circles is called a common tangent to the circles.

Let us consider Fig. 23·10. We note that in (*i*), we can draw four common tangents; in (*ii*), we can draw three common tangents; in (*iii*), we can draw one common tangent; in (*iv*), we have no common tangent and in (*v*), we have two common tangents.

In (*i*), let us denote the common tangents by l, m, n and p. l and m are called **direct common tangents.** n and p are called **transverse common tangents.**

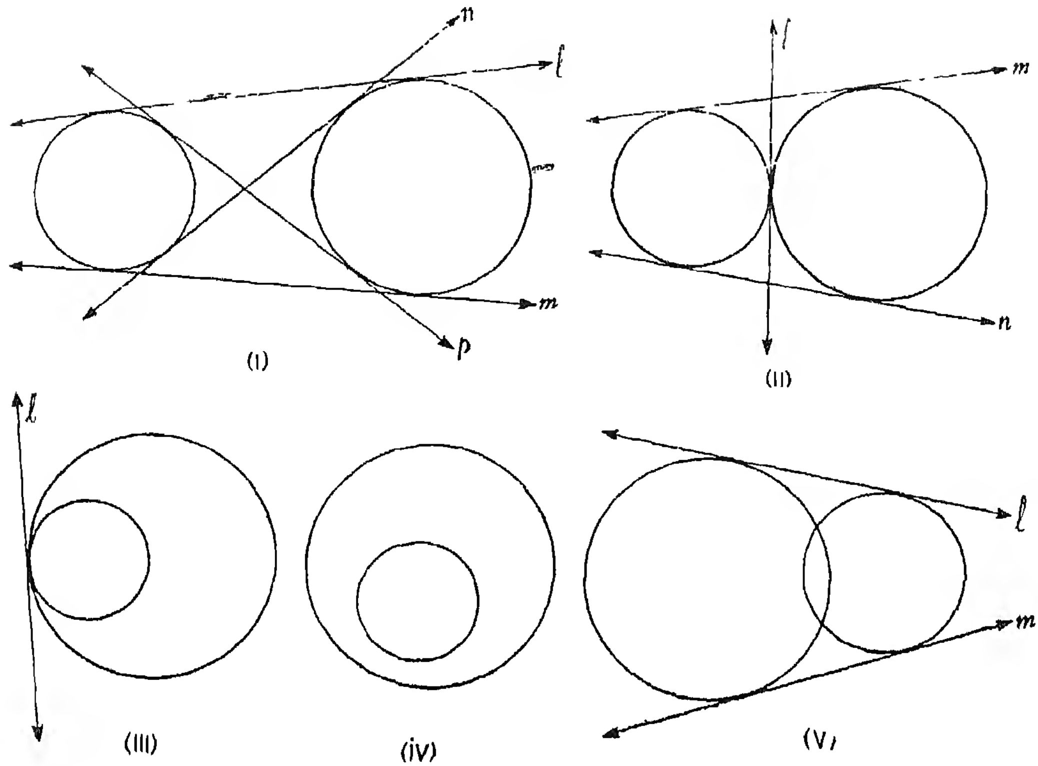


Fig. 23.10

Theorem 22 : **If two circles touch each other, the point of contact lies on the line joining their centres.**

Given : Two circles, with their respective centres A and B, touch each other at the point P. (See Fig. 23 11)

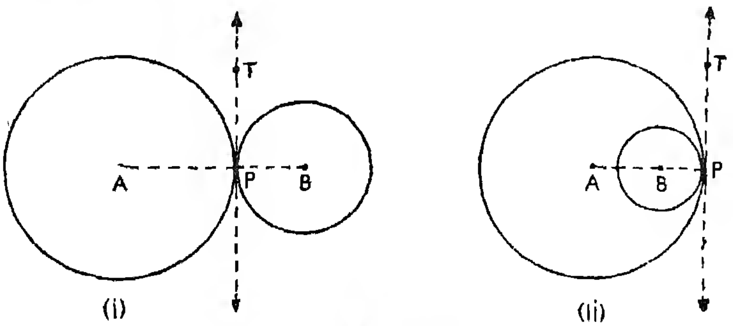


Fig. 23.11

To prove : P lies on the line AB

Construction : Draw the common tangent PT

Proof : $PA \perp PT$ (Theorem 20)

Also, $PB \perp PT$

But through a given point, we can draw only one perpendicular to the line PT.

Therefore, PA and PB must be the same line.

In other words, P lies on the line AB.

Q.E.D.

Note : The line joining the centres is the **line of symmetry** of the two circles.

Corollary : If r_1 and r_2 are the radii of the two circles and d the distance between their centres, then

In (*i*), $d = r_1+r_2$

And in (*ii*), $d = r_1-r_2$

Example 1 : Prove that the tangents at the end-points of a diameter of a circle are parallel.

Solution : Let l and m be the tangents to the circle at the end-points of a diameter AB. (See Fig. 23.12)

Then, $\angle x = 90°$

And, $\angle y = 90°$

$\therefore$ $\angle x = \angle y$

Whence, $l \parallel m$ ($\angle x$ and $\angle y$ are alternate angles)

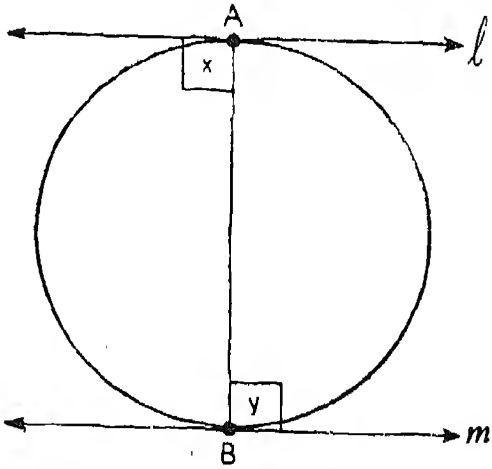


Fig. 23.12

Example 2 : If the sides of a quadrilateral touch a circle, prove that the sum of a pair of opposite sides is equal to the sum of the other pair.

Solution : Let $ABCD$ be a quadrilateral whose sides touch a circle at P, Q, R and S. (See Fig. 23.13)

We need to show that $AB+CD = AD+BC$

Since the lengths of the tangents from an external point to a given circle are equal, therefore

$$\left.\begin{aligned} AP &= AS \\ BP &= BQ \\ CR &= CQ \\ DR &= DS \end{aligned}\right\}$$

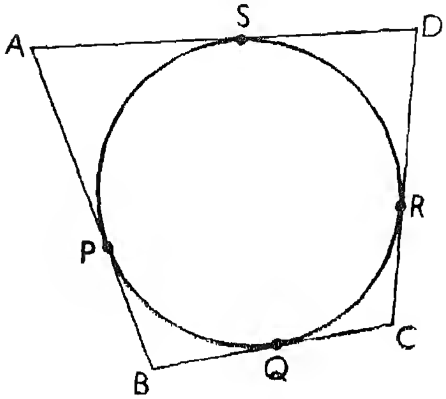


Fig. 23.13

Adding the above, we obtain

$$AB+CD = AD+BC$$

Example 3 : Given two circles with their respective centres A and B and radii r_1 and r_2 ($r_1 > r_2$), draw the direct common tangents.

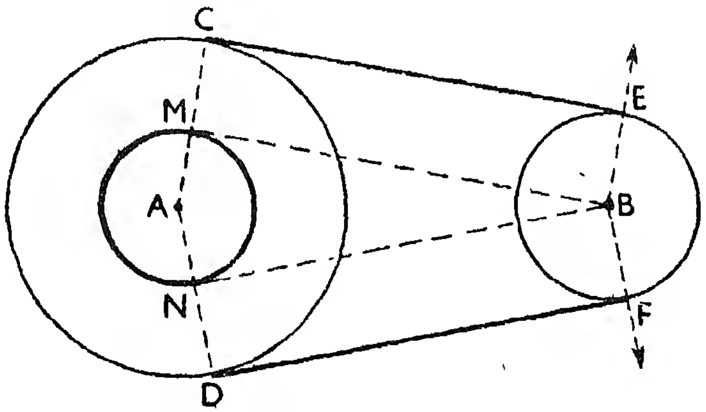


Fig. 23.14

Solution : The following steps are necessary :

Step 1 : Draw a circle with centre A and radius $r_1 - r_2$.

Step 2 : From B, draw tangents BM and BN to the circle drawn in Step 1.

Step 3 : Produce AM and AN to meet the circle, whose radius is r_1, at the points C and D.

Step 4 : From B, draw rays BE and BF parallel to AC and AD to intersect the circle, whose centre is B, at the points E and F respectively, as shown in Fig. 23.14.

Step 5 : Join C and E as also D and F.
CE and DF are the required direct common tangents

Example 4 : Given two circles with their respective centres A and B and radii r_1 and r_2, draw the transverse common tangents.

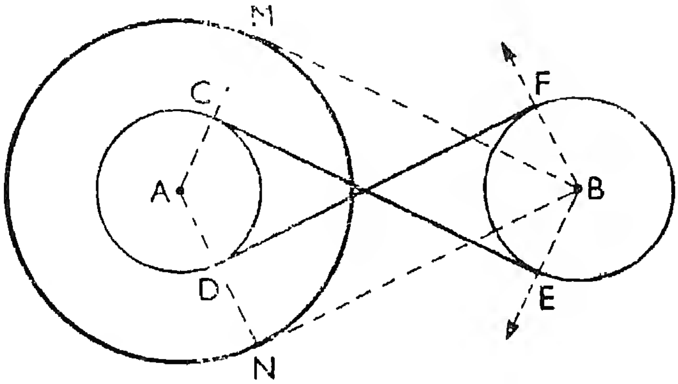


Fig. 23.15

Solution : The following steps are necessary :

Step 1 : With A as centre and radius $= r_1+r_2$, draw a circle

Step 2 : From B, draw tangents BM and BN to the circle drawn in Step 1.

Step 3 : Let AM and AN intersect the circle with radius r_1 at C and D.

Step 4 : From B, draw rays BE and BF parallel to AC and AD respectively as shown in Fig. 23.15.

Step 5 : Join C and E as also D and F.
CE and DF are the required transverse common tangents.

Note 1 : If we produce (if necessary) the tangents drawn in Examples 3 and 4, we will note that they meet at the line joining the centres of the two circles

Note 2 : The lengths of the segments CE and DF, i e., the lengths of the parts of the common tangents (direct or transverse) intercepted between their points of contact, are called the **lengths of the common tangents** (direct or transverse).

Example 5 : Draw a circle touching a given line at a given point and also touching a given circle.

Solution : Let l be a line and P be the given point on the given line l and let O be the centre of the given circle. (See Fig. 23.16)

We go through the following steps :

Step 1 : At P, draw $PQ \perp l$.

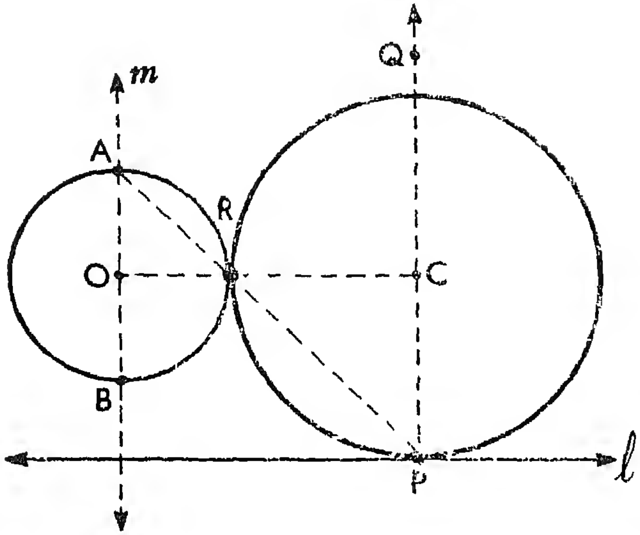


Fig. 23.16

Step 2 : Through O, draw $m \parallel PQ$. Let m intersect the given circle at the points A and B.

Step 3 : Join A and P. Let AP intersect the given circle at R.

Step 4 : Join O and R. Produce OR to meet PQ at C.

Step 5 : With C as centre and CP as radius, draw a circle.

This is the required circle.

(The reader is advised to draw the above circle in case $OP \perp l$.)

Exercise 23.1

1. Determine the length of a tangent to a circle with radius 5 cm, drawn from a point 13 cm from the centre of the circle.
2. Draw the incircle of a $\triangle ABC$ in which $BC=7$ cm, $CA=5$ cm and $AB = 6$ cm.

3. Draw the three excircles of a $\triangle PQR$ in which $PQ = 5$ cm, $\angle P = 70°$ and $\angle Q = 60°$.

4. In Fig. 23.17, three circles with equal radii touch each other externally.

   Show that the triangle formed by joining their centres is equilateral.

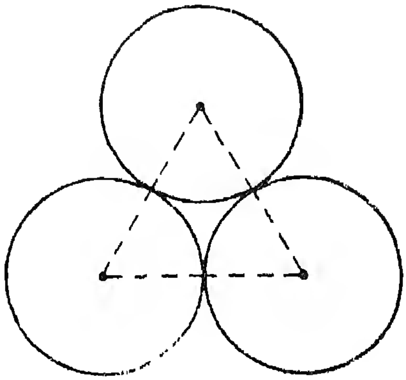


Fig. 23.17

5. Prove that the two tangents, drawn to a circle from a point outside it, subtend equal angles at the centre.

6. Two circles touch each other externally. Show that the lengths of the tangents, drawn to the two circles from any point on the common tangent, are equal.

7. Draw a circle of radius 3 cm. Take a point P, 5 cm from its centre. From P, draw a tangent to the circle and determine its length. Verify the length by actual measurement.

8. In Fig. 23.18, AB and AC are the tangents drawn from an external point A to a circle whose centre is O.

   Show that OA is the perpendicular bisector of BC.

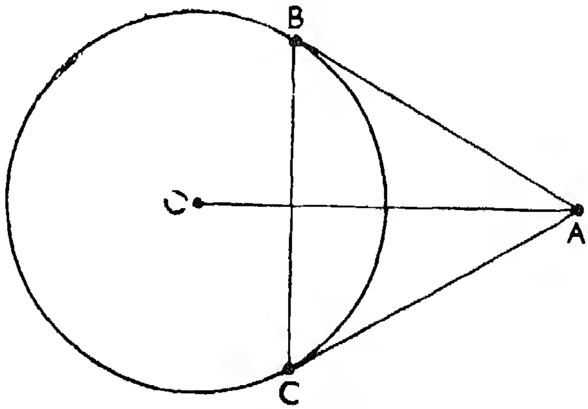


Fig. 23.18

9. In Fig. 23.19, a circle touches the side BC of a $\triangle ABC$ at P and AB and AC produced at Q and R respectively.

Prove that AQ is half the perimeter of $\triangle ABC$.

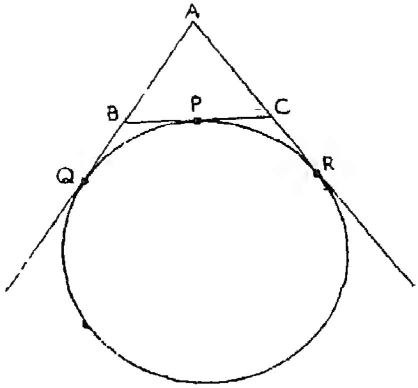


Fig. 23.19

10. In Fig. 23.20, the incircle of $\triangle ABC$ touches the sides BC, CA and AB at D, E and F respectively.

Show that

$AF+BD+CE=AE+CD+BF$

$= \frac{1}{2}$ (Perimeter of $\triangle ABC$)

Show further that if a, b, c are the lengths of the sides BC, CA, AB and s the semi-perimeter of $\triangle ABC$, then $AF = s-a$, $BD = s-b$ and $CE = s-c$.

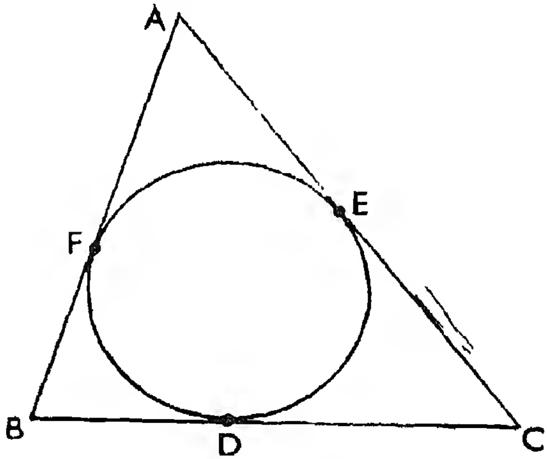


Fig. 23.20

11. If all the sides of a parallelogram touch a circle, show that the parallelogram is a rhombus.
12. P is the mid-point of an arc APB of a circle. Show that the tangent at P is parallel to the chord AB.
13. A chord AB of the larger of the two concentric circles is tangent to the smaller circle at the point P. Show that P is the mid-point of AB.
14. Given a $\triangle ABC$ in which $AB = AC$. The incircle of the triangle touches the side BC at D. If O is the incentre, show that DO is the perpendicular bisector of BC.

15. In Fig. 23.21, PAB and PCD are direct common tangents to the two circles from an external point P.

Prove that $AB = CD$

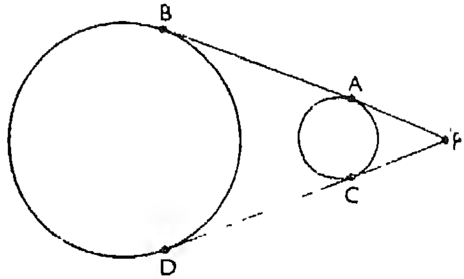


Fig. 23.21

16 Show that the direct common tangents of two congruent circles are equal and parallel

17. Given two circles with their respective centres A and B and equal radii, draw their direct common tangents.

18. In Fig. 23 22, AB and CD are transverse common tangents to the two circles.

Prove that $AB = CD$

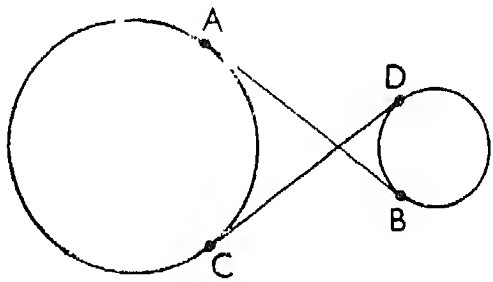


Fig. 23.22

19. Draw two circles of radii 4 cm and 2 cm in such a way that the distance between their centres is 9 cm. Further, draw the direct common tangents to these circles and measure their lengths. Are they equal ?

20. Draw two circles of radii 4 cm and 3 cm in such a way that the distance between their centres is 9 cm. Draw the transverse common tangents to these circles and measure their lengths. Are they equal ?

21. Show that the tangent at the point of contact of two circles bisects their direct common tangents.

22. In Fig. 23 23, a circle touches the sides of a quadrilateral $ABCD$ at the points P, Q, R and S respectively.

Show that the angles subtended at the centre by a pair of opposite sides are supplementary.

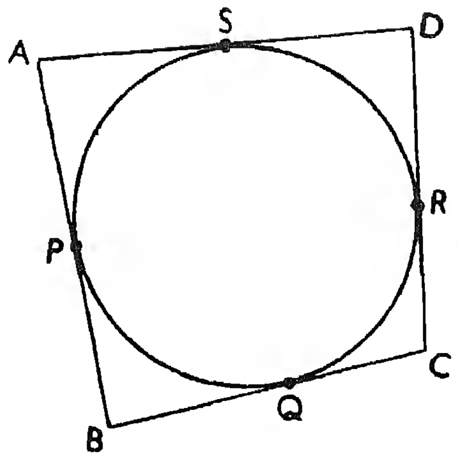


Fig. 23.23

23. From a point P, tangents PA and PB are drawn to a circle whose centre is O as shown in Fig. 23.24. If the distance of P from O is equal to the diameter, prove that ABP is an equilateral triangle.

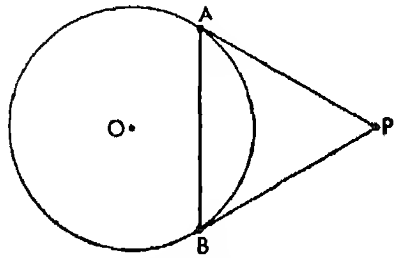


Fig. 23.24

24 Draw a circle touching a line at a given point and passing through another point not lying on the line.

25. Let $AB = 4$ cm. With centre A, draw a circle C of radius 2 cm. Draw another circle of radius 3 cm passing through B and touching the circle C.

26. Given a triangle ABC in which $BC = 6$ cm, $CA = 5$ cm and $\angle C = 80°$, draw a circle touching the side CA at C and passing through B.

27. Given a circle, a point P on it and a line l not intersecting the circle, draw a circle touching l and also touching the given circle at P.

*28. Given a circle of radius 3·5 cm and a line l, 5 cm from its centre, draw circles of radii 2.5 cm each touching the given circle and the line l.

23.4. Angles in the Alternate Segments

Let AB be a chord of a circle and SAT a tangent at A. S and T are points lying on opposite sides of A.

Let P and Q be two points on the circle on opposite sides of the line AB as shown in Fig. 23.25 We say that the segment formed by the arc APB is **alternate** to $\angle BAT$ and the segment formed by the arc AQB is **alternate** to $\angle BAS$.

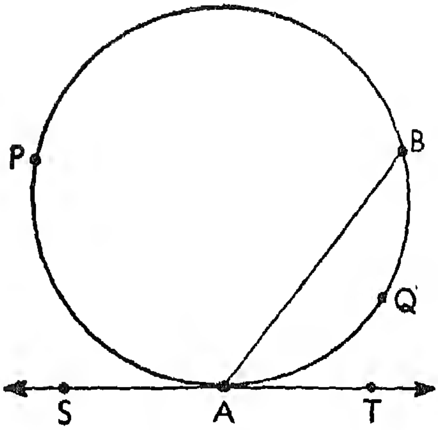


Fig. 23.25

Theorem 23 : **If a line touches a circle and from the point of contact a chord is drawn, the angles which this chord makes with the given line are equal respectively to the angles formed in the corresponding alternate segments.**

Given : Line SAT is a tangent at A and AB is a chord through A. P and Q are points on the circle on opposite sides of the line AB as shown in Fig. 23.26.

To prove : (*i*) $\angle BAT = \angle APB$

(*ii*) $\angle BAS = \angle AQB$

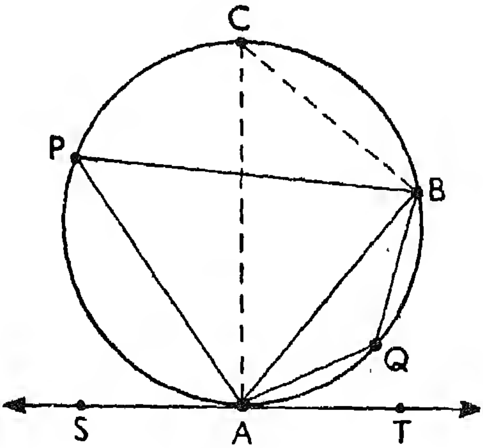


Fig. 23.26

Construction : Through A, draw a diameter AC of the circle. Join B and C.

Proof :	$\angle ABC = 90°$	(Angle in a semi-circle)
$\therefore$	$\angle BAC + \angle ACB = 90°$	(*i*)
But,	$\angle ACB = \angle APB$	(Angles in the same segment)
$\therefore$	$\angle BAC + \angle APB = 90°$	[From (*i*)] (*ii*)
Also,	$\angle CAT = 90°$	
i.e.,	$\angle BAC + \angle BAT = 90°$	(*iii*)
Therefore,	$\angle BAT = \angle APB$	[From (*ii*) and (*iii*)]

Q.E.D. (i)

Now,	$\angle APB + \angle AQB = 180°$	(Why ?)
$\therefore$	$\angle BAT + \angle AQB = 180°$	(*iv*)
Also,	$\angle BAT + \angle BAS = 180°$	(*v*)
Therefore,	$\angle BAS = \angle AQB$	[From (*iv*) and (*v*)]

Q.E.D. (ii)

Example 1 : PQ and PR are equal chords of a circle. (See Fig 23.27) Prove that the tangent SPT to the circle at P is parallel to the chord QR.

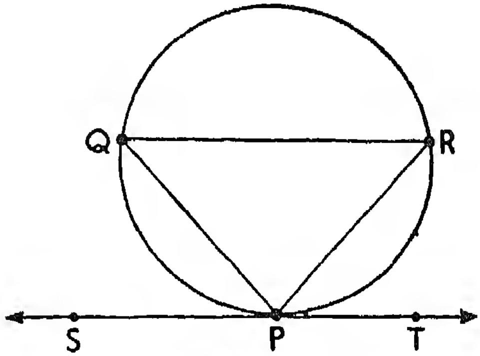


Fig. 23.27

Solution :	$PQ = PR$	(Given)	
$\therefore$	$\angle PQR = \angle PRQ$		(*i*)
Also,	$\angle SPQ = \angle PRQ$	(Theorem 23)	(*ii*)
$\therefore$	$\angle PQR = \angle SPQ$	[From (*i*) and (*ii*)]	
Whence,	$SPT \parallel QR$	(Why ?)	

Example 2 : Two circles intersect each other at the points A and B. CD is a direct common tangent. (See Fig 23.28) Prove that the angles subtended by the segment CD at A and B are supplementary.

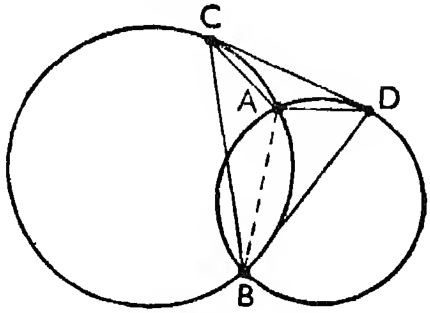


Fig. 23.28

Solution : We need to show that

$$\angle CAD + \angle CBD = 180^\circ$$

Now, $\angle ABC = \angle ACD$ } (Theorem 23)

And, $\angle ABD = \angle ADC$ }

$\therefore$ $\angle ABC + \angle ABD = \angle ACD + \angle ADC$ (*i*)

Adding $\angle CAD$ to both sides of (*i*) we obtain

$$\angle CAD + \angle CBD = 180^\circ$$

Example 3 : Given a line-segment, construct on it a segment of a circle such that the angle in the segment is equal to a given angle.

Solution : Two cases arise.

Case I : When the given angle is a right angle

Let AB be a given line-segment and let O be the mid-point of AB. (See Fig 23.29)

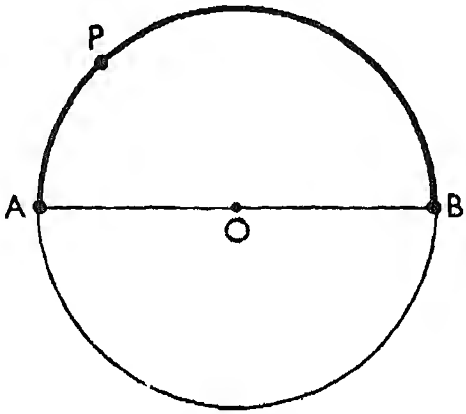


Fig. 23.29

The following steps are necessary :

Step 1 · With O as centre and OA as radius, draw a semi-circle APB

The segment formed by the semi-circle APB is the required segment.

Case II : When the given angle is not a right angle

Let x be a given angle. [See Fig. 23.30 (*i*)]

The following steps are necessary :

Step 1 : On the given line-segment AB, draw an angle BAC equal to $\angle x$.

Step 2 : Draw $AP \perp AC$. Let AP meet the perpendicular bisector of AB at O.

Step 3 : With O as centre and OA as radius, draw an arc ADB as shown in Fig. 23 30 (*ii*).

The segment of the circle formed by the arc ADB is the required segment

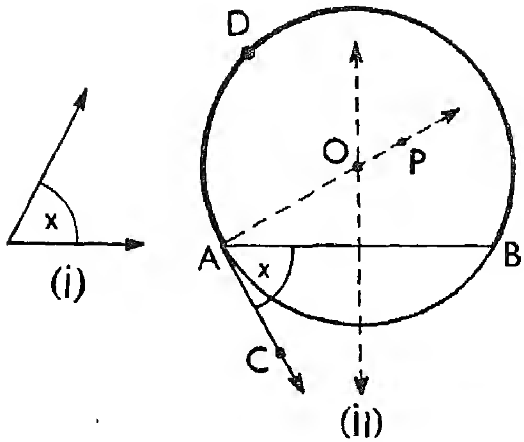


Fig. 23.30

Exercise 23.2

1. In Fig 23.31, $KLMN$ is a cyclic quadrilateral and PQ is a tangent to the circle at K. If LN is a diameter of the circle, $\angle KLN = 30°$ and $\angle MNL = 60°$, determine

   (i) $\angle QKN$ (ii) $\angle PKL$

   (iii) $\angle LKN$ (iv) $\angle LMN$

   (v) $\angle MLN$

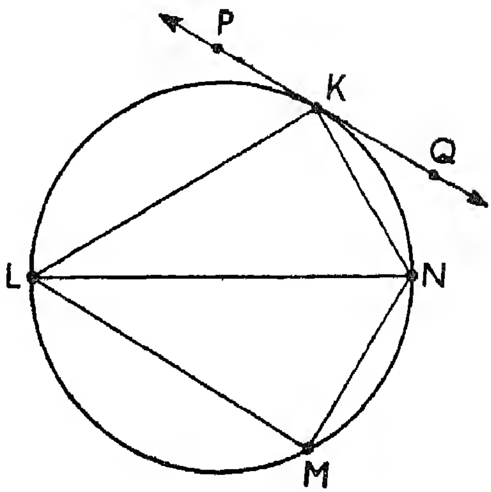


Fig. 23.31

2. In Fig. 23.32, AB is a chord of the circle and $\angle BAQ = \angle ACB$. (C is a point, on the circle, lying on a side of AB opposite to Q)

   Prove that AQ is a tangent to the circle at A.

   (**Hint** : Prove by contradiction)

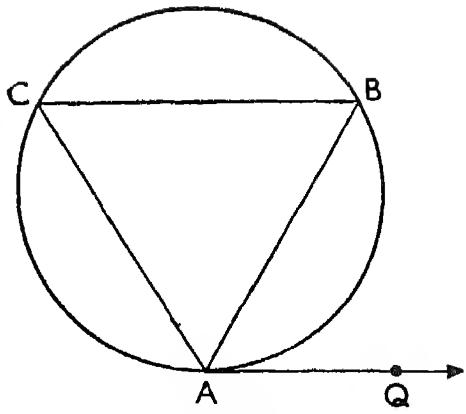


Fig. 23.32

3. In Fig. 23.33, CD is a tangent to the circumcircle of a $\triangle ABC$ at C, intersecting AB produced at D.

   Show that $\triangle DBC \sim \triangle DCA$

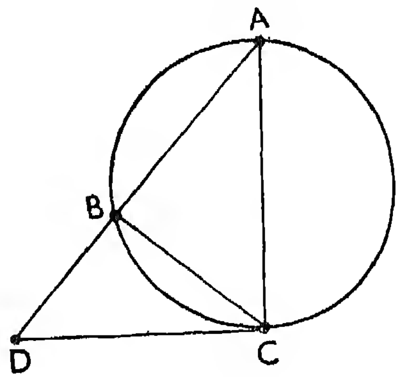


Fig. 23.33

4. In Fig. 23.34, DE is a tangent to the circumcircle of $\triangle ABC$ at the vertex A such that $DE \parallel BC$.

   Show that $AB = AC$

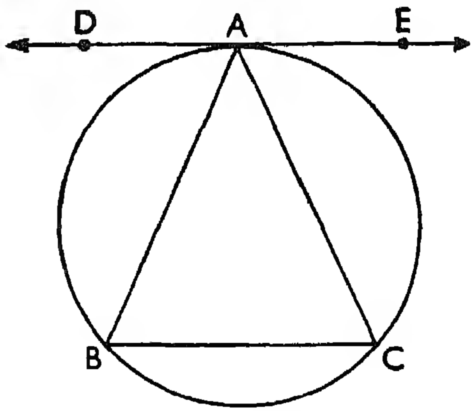


Fig. 23.34

5. In Fig. 23.35, two circles C_1 and C_2 intersect at A and B. At A, tangents are drawn to C_1 and C_2 to intersect C_2 at C and C_1 at D.

Prove that $\triangle ABC \sim \triangle DBA$

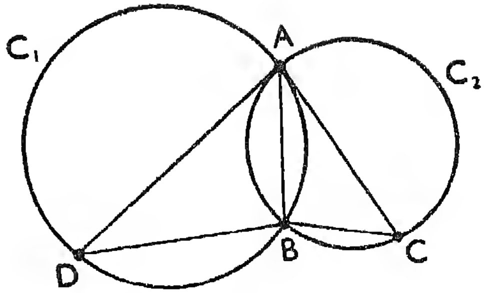


Fig. 23.35

6. In Fig 23.36, TP is a tangent and PAB is a secant to a circle. If the bisector of $\angle ATB$ intersects AB at M, show that

(i) $\angle PMT = \angle PTM$

(ii) $PT = PM$

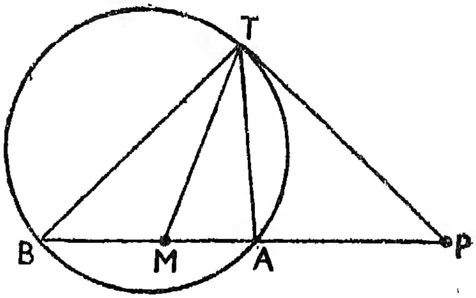


Fig. 23.36

7. In Fig. 23.37, two circles touch each other at P. APC and BPD are lines through P that meet the two circles in A, B and C, D respectively.

Show that

(i) $\triangle PAB \sim \triangle PCD$

(ii) $AB \parallel CD$

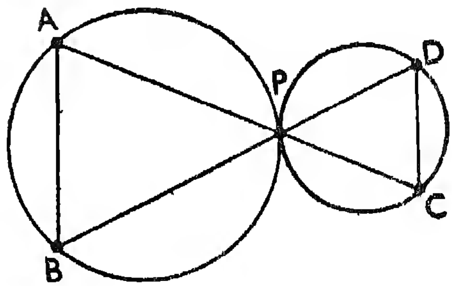


Fig. 23.37

8. In Fig 23.38, two circles touch each other at P. Two lines PCA and PDB meet the circles in C, D and A, B respectively

Prove that $AB \parallel CD$

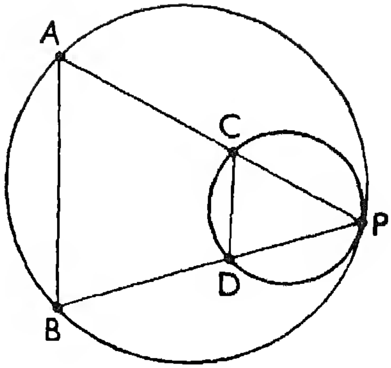


Fig. 23.38

*9. In Fig. 23.39, PA and PB are tangents to a circle drawn from a point P outside the circle and M is the mid-point of the minor arc AB.

Show that AM is the bisector of $\angle PAB$.

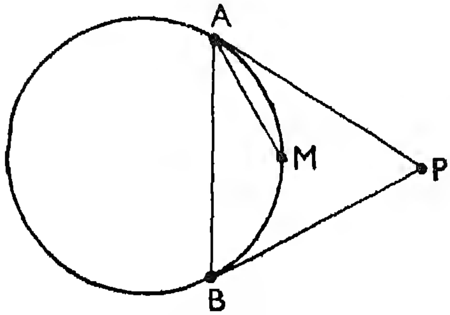


Fig. 23.39

10. In Fig. 23.40, two circles, one of which passes through the centre O of the other, intersect at A and B.

If AT is a tangent to the circle, that passes through O, at A, show that AO bisects $\angle BAT$.

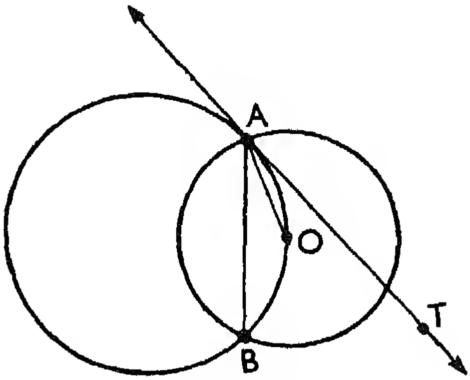


Fig. 23.40

*11. In Fig 23 41, ABC is a triangle right angled at C. A circle, with BC as diameter, is drawn to intersect the side AB at T.

If tangent to the circle at T intersects CA at M, show that M is the mid-point of side CA.

(**Hint** : Join T and C)

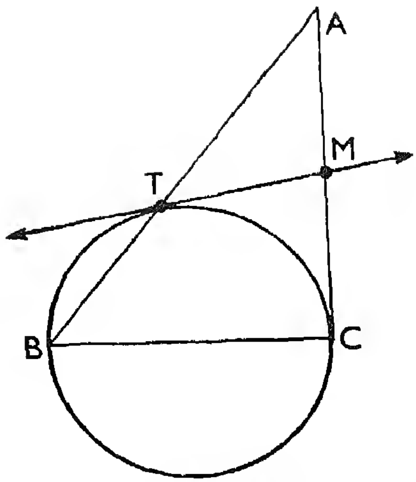


Fig. 23.41

*12. Two circles intersect at A and B. Through a point T on one of the circles, segments TAC and TBD are drawn such that C and D lie on the other circle as shown in Fig. 23.42.

Show that the chord CD is parallel to the tangent TM.

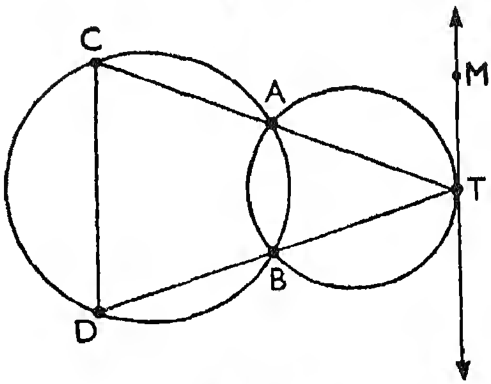


Fig. 23.42

13. In Fig. 23.43, AT is the tangent to the circumcircle of $\triangle ABC$ at the vertex A. A line parallel to AT intersects the sides AB and AC at the points D and E respectively.

Prove that

(i) $\triangle ABC \sim \triangle AED$
(ii) $AB.AD = AC.AE$

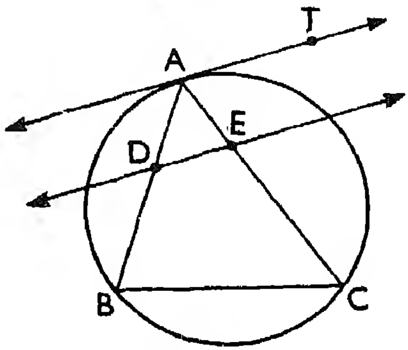


Fig. 23.43

14. Draw a line-segment of length 6 cm. On it, construct the segment of a circle such that any angle in the segment is 110°.
15. Given a point on a circle, draw the tangent to the circle at the given point without finding the centre of the circle.
(Hint : Use Theorem 23)

23.5. Rectangle Theorems—Intersecting Chords

Let us consider a segment AB and a point E on it. (See Fig. 23.44) We recall that $AE \times EB$ represents the area of the rectangle whose sides are equal to AE and EB. We call this the area of the **rectangle contained** by the parts of AB formed by E.

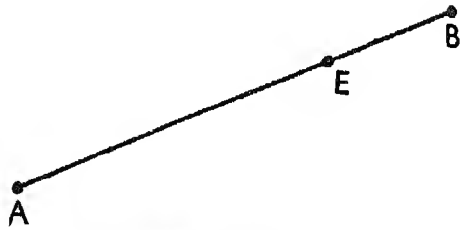


Fig. 23.44

Theorem 24 : **If two chords of a circle intersect inside the circle, the rectangle contained by parts of one chord is equal in area to the rectangle contained by the parts of the other.**

Given : AB and CD are the two chords of a circle intersecting at E. (See Fig. 23.45)

To prove : $AE \times EB = CE \times ED$

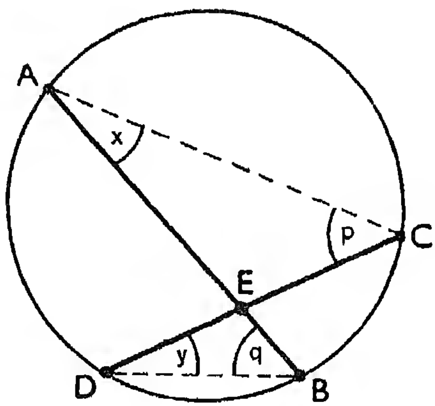


Fig. 23.45

Construction : Join A and C as also B and D.

Proof : In $\triangle$s AEC and DEB

$$\left.\begin{array}{r} \angle x = \angle y \\ \text{and} \quad \angle p = \angle q \end{array}\right\} \quad \text{(Angles in the same segment)}$$

$\therefore$ $\triangle AEC \sim \triangle DEB$ (AAA)

Whence, $\dfrac{AE}{ED} = \dfrac{CE}{EB}$

i.e., $AE \times EB = CE \times ED$

Q.E.D.

Theorem 25 : **If two chords of a circle intersect when produced, the rectangle contained by the two parts into which one chord is divided externally is equal in area to the rectangle contained by the two parts of the other.**

Given : AB and CD are two chords of a circle. When produced, the chords intersect at E. (See Fig. 23.46)

To prove : $EA \times EB = EC \times ED$

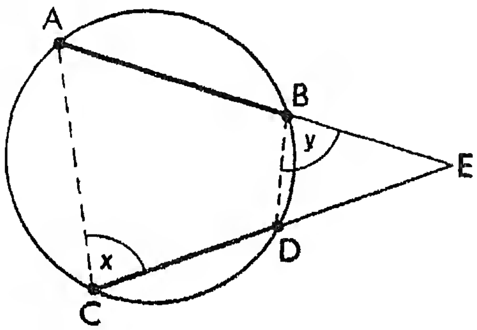


Fig. 23.46

Construction : Join A and C as also B and D.

Proof : In $\triangle$s EAC and EDB

$\angle x = \angle y$ (Exterior angle of a cyclic quadrilateral)

$\therefore$ $\triangle EAC \sim \triangle EDB$ (AAA)

Whence, $\dfrac{EA}{ED} = \dfrac{EC}{EB}$

i.e., $EA \times EB = EC \times ED$

Q.E.D.

Corollary : In Theorem 25, if C coincides with D, it is obvious that the secant EDC will become a tangent at D. Consequently, we obtain

$$EA \times EB = ED \times ED$$

i.e., $$EA \times EB = ED^2$$

In other words,

If from an external point, a secant is drawn to a circle, the rectangle contained by the two parts into which the chord contained in the secant is divided externally by the given point is equal in area to the square on the tangent to the circle from the given point.

Example 1 : In Fig. 23.47, $PA = 10$ cm, $PB = 4$ cm and $PC = 8$ cm. Determine PD.

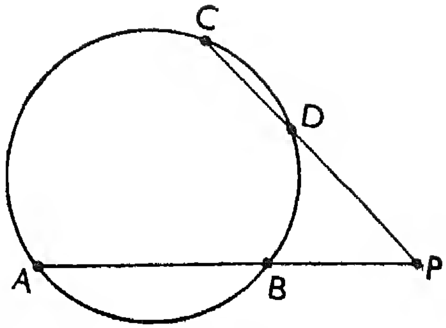


Fig. 23.47

Solution : $PC \times PD = PA \times PB$

i.e., $8\,(PD) = 40$

Whence, $PD = 5$

PD is, therefore, 5 cm

Example 2 : In Fig. 23 48, $AP = CP$.

Show that $AB = CD$

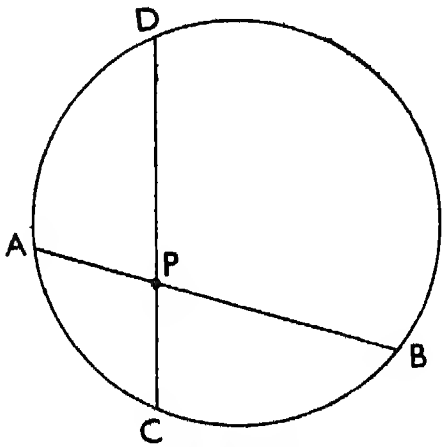


Fig. 23.48

Solution : $AP \times BP = CP \times DP$

i.e., $BP = DP$ (Since $AP = CP$)

$\therefore$ $AP + BP = CP + DP$

Whence, $AB = CD$

Example 3 : Construct a square equal in area to a given rectangle.
Solution : Let $ABCD$ be the given rectangle. (See Fig. 23.49)

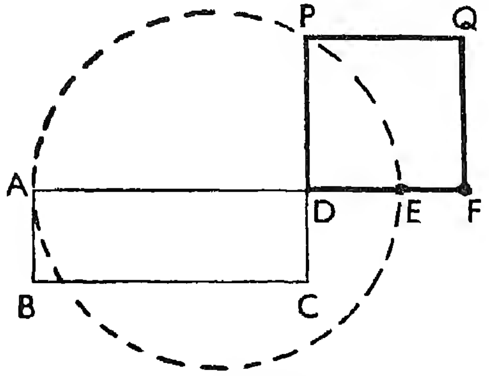


Fig. 23.49

The following steps are necessary :

Step 1 : Produce AD to E such that $DE = DC$.

Step 2 : With AE as diameter, draw a circle to intersect CD produced in P.

Step 3 : The square with side DP is equal in area to the rectangle $ABCD$. $DPQF$ is the required square.

(The reader is advised to furnish a proof to this construction)

What if we wanted to construct a square equal in area to a given polygon ? We proceed through the following steps :

Step 1 : Construct a triangle equal in area to the given polygon. (See Section 14.2, Part I of this book.)

Step 2 : Construct a rectangle equal in area to this triangle.

Step 3 : Finally, construct a square equal in area to this rectangle (See Example 3 above)

Exercise 23.3

1. In Fig. 23.50

(*i*) $AP = 8$ cm, $CP = 6$ cm and $PD = 4$ cm. Determine PB.

(*ii*) $AB = 12$ cm, $AP = 2$ cm and $PD = 4$ cm. Determine CP.

(*iii*) $AP = 6$ cm, $PB = 5$ cm and $CD = 13$ cm. Determine CP.

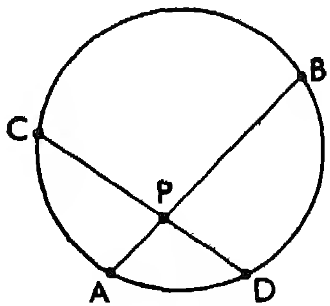


Fig. 23.50

2. In Fig 23.51

(i) $PA = 16$ cm, $PC = 10$ cm and $PD = 8$ cm. Determine AB.

(ii) $PC = 15$ cm, $CD = 7$ cm and $PA = 12$ cm Determine AB.

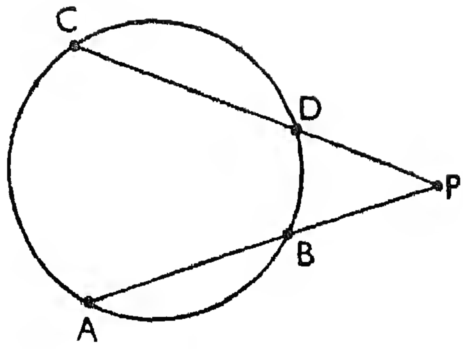


Fig. 23.51

3. In Fig 23.52, PBA is a secant and PT is a tangent to the circle such that $PT = 2PB$.

Determine PT if $AB = 9$ cm.

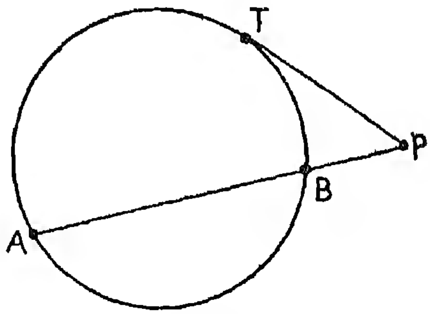


Fig. 23.52

4. In Fig. 23.53, if $AP = CP$ prove that $AB = CD$.

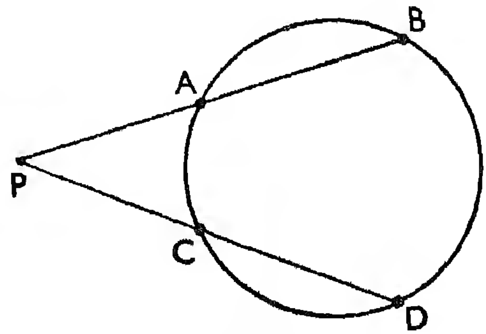


Fig. 23.53

5. In Fig. 23.54, ABC and DBC are two right triangles with a common hypotenuse BC and with their sides AC and DB intersecting at P.

Prove that

$$AP \times PC = BP \times PD$$

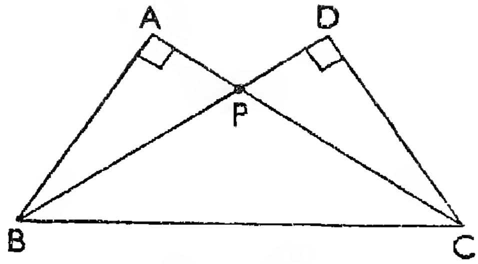


Fig. 23.54

6. PT is a tangent at T and PBA is a secant to a circle whose centre is O. If $\angle APT = 90°$, show that

$$PA^2 + PB^2 + 2PT^2 = 4\,(OT^2)$$

7. In Fig. 23.55, AB is a direct common tangent to the two circles intersecting at the points M and N.

Prove that the line containing the common chord MN bisects AB.

(**Hint :** $PM \times PN = PA^2$, etc.)

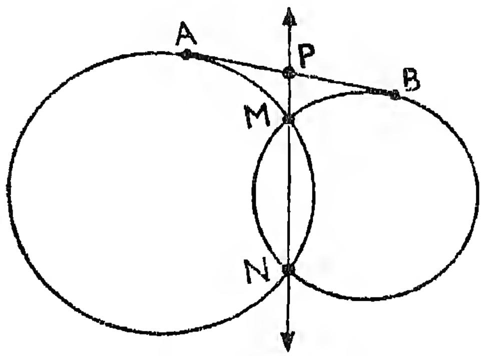


Fig. 23.55

8. From the vertices B and C of a $\triangle ABC$, perpendiculars BD and CE have been drawn to their opposite sides to intersect each other at P. (See Fig. 23.56)

Prove that

$$BP \times PD = EP \times PC$$

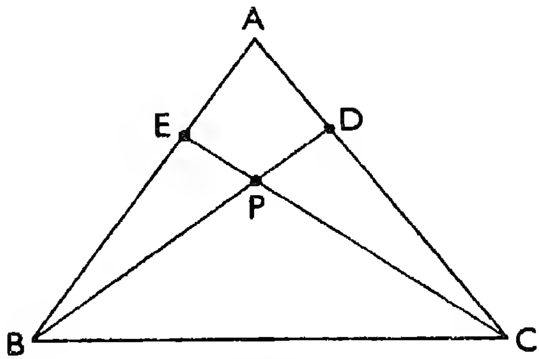


Fig. 23.56

9. In Fig. 23.57, ABC is a right triangle with $\angle A = 90°$ AD is perpendicular, drawn from A, to the hypotenuse BC

Prove that

(*i*) circle, with diameter AC, passes through D.

(*ii*) AB is the tangent to this circle.

(*iii*) $BC \times BD = AB^2$.

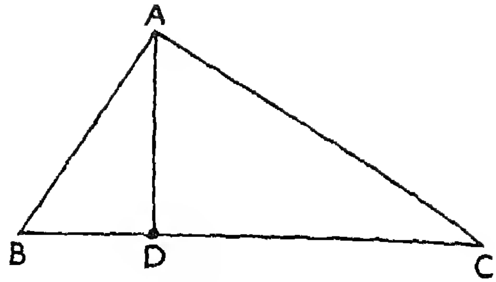


Fig. 23.57

10. In Fig 23.58, AB and CD are segments that intersect at P such that $AP \times PB = CP \times PD$. If the circle passing through A, B and C intersects CD at E, show that

(*i*) $CP \times PD = CP \times PE$

(*ii*) D must coincide with E.

(*iii*) A, B, C and D are concyclic.

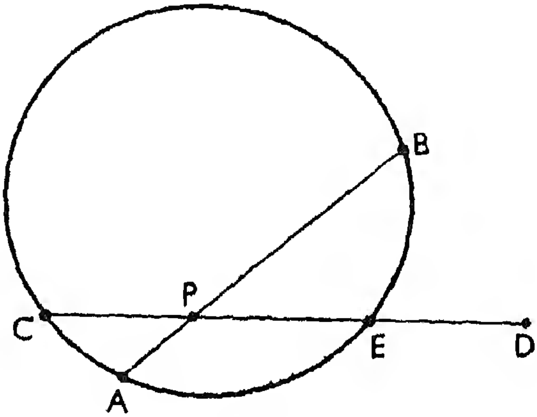


Fig. 23.58

(The above result can alternatively be stated as
If two segments intersect each other internally in such a way that the rectangle contained by the parts so formed of one segment is equal in area to the rectangle contained by the parts of the other, the four end-points of the two segments are concyclic.)

11 In Fig. 23.59, AB and CD are segments that intersect, when produced, at P such that

$$PA \times PB = PC \times PD$$

If the circle passing through points A, B and C intersects CD at E, show that

(*i*) D must coincide with E.

(*ii*) A, B, C and D are concyclic.

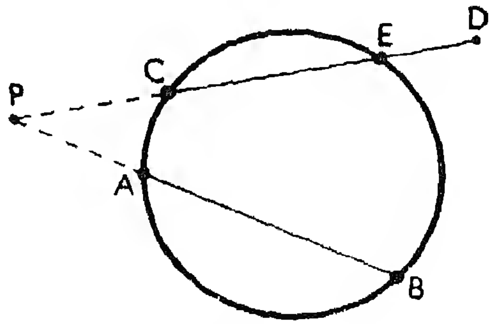


Fig. 23.59

(This result can alternatively be stated as

If two segments intersect, when produced, in such a way that the rectangle contained by the parts of one segment formed due to its external division by the point of intersection is equal in area to the rectangle contained by the parts of the other, the four end-points of the two segments are concyclic.)

12. In Fig. 23.60, AB is a segment. From a point P on BA produced, a segment PT is drawn such that $PA \times PB = PT^2$. If the circle through A, B and T intersects PT at S, show that

(i) $PS \times PT = PT^2$

(ii) T must coincide with S.

(iii) PT is a tangent to the circle.

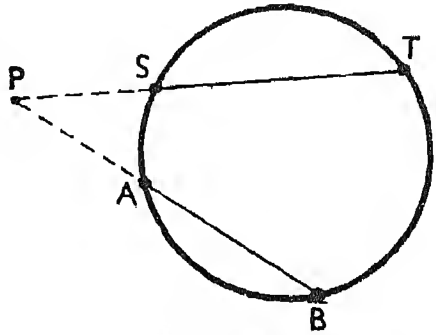


Fig. 23.60

(The above result can alternatively be stated as

If a chord of a circle is divided externally by a point and if a segment is drawn joining this point and a point on the circle such that the square on this segment is equal in area to the rectangle contained by the parts of the chord formed due to external division, the segment is the tangent to the circle.)

13. In Fig. 23.61, OD and OAB are two segments such that
$OD^2 = OA \times OB$.
Prove that $\triangle OAD \sim \triangle ODB$

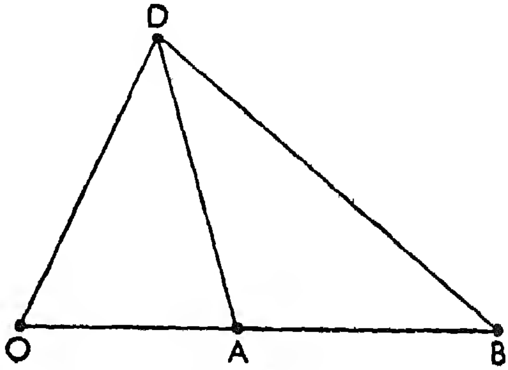


Fig. 23.61

14. In Fig. 23.62, from a point P on the common chord MN produced, secants PAB and PCD are drawn, one to each circle.

Show that

(i) $PA \times PB = PC \times PD$

(ii) A, B, C and D are concyclic.

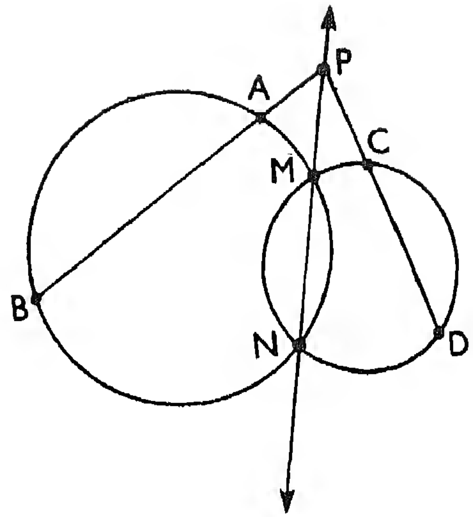


Fig. 23.62

15. In Fig. 23.63, points A and B lie on one circle and points C and D on the other such that segments AC and BD pass through one of the points of intersection M of the two circles.

If BM is a diameter of one circle and CM of the other, prove that

(i) Points A, B, C and D are concyclic.

(ii) $AM \times CM = BM \times DM$

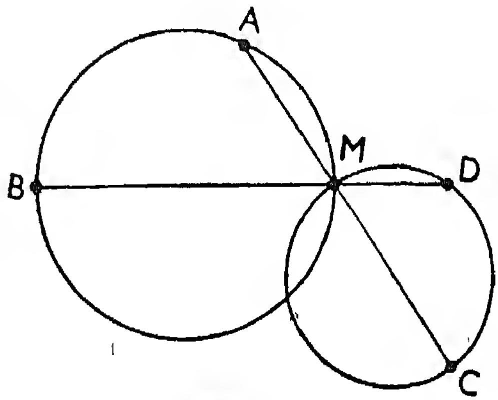


Fig. 23.63

16. Divide a line-segment AB **internally** so that the rectangle contained by the two parts is equal in area to the square on another line-segment CD.
17. Divide a line-segment AB **externally** so that the rectangle contained by the two parts is equal in area to the square on another line-segment CD.
18. Draw a rectangle $ABCD$ of sides 7 cm and 4 cm. Construct a square equal in area to this rectangle.
19. Draw a circle touching a given line and passing through two given points (not lying on the line).

CHAPTER XXIV

MENSURATION

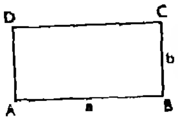
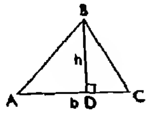
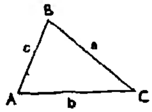
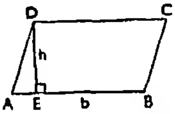
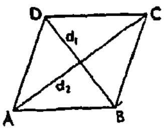
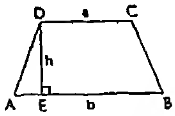
24.1. Introduction

The word 'mensuration' is derived from the Greek word mēnsūrātiō', meaning 'to measure' and refers to that branch of geometry which deals with the measurement of length, area or volume.

The reader is already familiar with the lengths and with the areas of certain simple geometric figures, such as, rectangles, triangles, parallelograms, etc. (See Chapter XIV, Part I of this book) Depending upon the unit of length, we have a corresponding unit of area. We give below some widely-used units of area and their equivalents :

100 sq. millimetres (sq. mm)	**= 1 sq. centimetre (sq. cm)**
10,000 sq. centimetres (sq. cm)	**= 100 sq. decimetres (sq. dm)**
	= 1 sq. metre (sq. m)
100 sq. metres (sq. m)	**= 1 are**
10,000 sq. metres (sq. m)	**= 100 ares = 1 hectare (ha)**
10,00,000 sq. metres (sq. m)	**= 100 hectares (ha)**
	= 1 sq. kilometre (sq. km)

We summarise below the formulae of areas of certain plane geometric figures :

Name of the geometric figure	*Area (A)*	*Figure*	*Reference Part I of this book Page)*	*Remarks*
Rectangle	$a \times b$		249	—
Triangle	$\frac{1}{2} b \times h$		254	—
Triangle	$\sqrt{s(s-a)(s-b)(s-c)}$			$s = \frac{a+b+c}{2}$
Parallelogram	$b \times h$		252	
Rhombus	$\frac{1}{2} d_1 \times d_2$			d_1 and d_2 are the lengths of the diagonals
Trapezium	$\frac{1}{2}(a+b) \times h$		256	

(The reader is advised to note that a rhombus is a particular type of a parallelogram and, therefore, an alternative formula for the area of a rhombus could be $A = b \times h$, where b is a side and h is the altitude.)

Example 1 : The length of a rectangular floor is twice its width. If the length of a diagonal is $9\sqrt{5}$ m, find the perimeter and the area of the floor.

Solution : Let us denote the floor by $ABCD$ and its width by x. (See Fig. 24.1) Now, in $\triangle CAB$

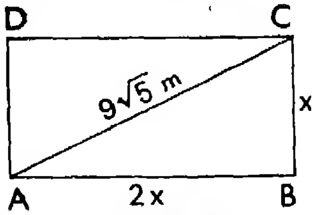


Fig. 24.1

$$4x^2 + x^2 = 405 \quad \text{(Why ?)}$$

Whence, $x = 9$

Thus, the dimensions of the floor are 18 m and 9 m.

$$\text{Perimeter} = 2(\text{Length} + \text{Width})$$
$$= 54 \text{ m}$$

$$\text{Area} = 162 \text{ sq m}$$

Hence, the perimeter of the rectangular floor is 54 m, while its area is 162 sq m.

Example 2 : In a triangle ABC, $AC = 8$ cm, $BC = 6$ cm and $\angle ACB = 30°$. Determine the area of the triangle.

Solution : Let us denote the altitude corresponding to the side BC by AD. (See Fig. 24.2) From our knowledge of trigonometry, we note that

$$\frac{AD}{AC} = \sin 30°$$

Whence, $AD = 4$ cm

Thus, area of $\triangle ABC = \frac{1}{2}(6)(4)$ or 12 sq. cm.

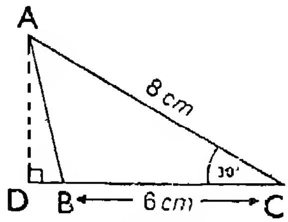


Fig 24.2

Example 3 : In a $\triangle ABC$, $a = 13$ cm, $b = 14$ cm and $c = 15$ cm. Determine the area of the triangle and its altitude corresponding to AC.

Solution : We first determine s

$$s = \frac{a+b+c}{2} = 21 \text{ cm}$$

$$\therefore \quad \text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{21(21-13)(21-14)(21-15)}$$

$$= \sqrt{21 \times 8 \times 7 \times 6}$$

$$= 84$$

The area of the $\triangle ABC$ is, therefore, 84 sq. cm.

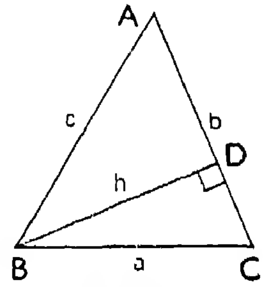


Fig. 24.3

Let us denote by h, the altitude corresponding to AC. (See Fig. 24.3)

Then, $\frac{1}{2} b \times h = 84$

Whence, $h = 12$. (Since $b = 14$ cm)

Hence, the altitude corresponding to the side $AC = 12$ cm.

Example 4 : The area of a parallelogram $ABCD$ is 120 sq. cm. If the side $AB = 15$ cm, find the distance between AB and DC. Further, if $\angle CDA = 150°$, find the perimeter of the parallelogram.

Solution : Let us denote by h, the distance between AB and DC. (See Fig. 24.4)

Then, Area $= b \times h$

i e., $120 = 15\,h$

Whence, $h = 8$

Thus, the distance between AB and DC is 8 cm.

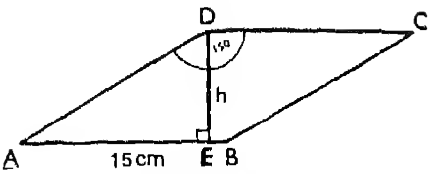


Fig. 24.4

Now, $\angle CDA + \angle DAB = 180°$ (Why ?)

$\therefore$ $\angle DAB = 30°$

Whence, $DE = DA$ (sin 30°) (Why ?)

i.e , $DA = 16$ cm

The perimeter of the parallelogram, therefore, is 62 cm.

Example 5 : A sheet is in the form of a rhombus, whose side is 5 m and one of the diagonals is 8 m. Find the cost of painting both the surfaces at the rate of Rs 3 50 per sq. m.

Solution : Let us denote the rhombus by $ABCD$ and the point where its diagonals intersect by O. (See Fig. 24.5)

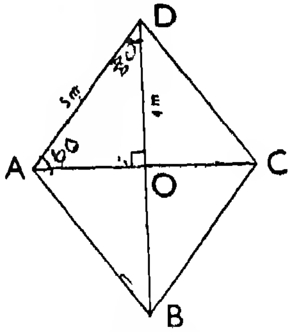


Fig. 24.5

Then, $OD = 4$ m (Why ?)

Now, $AO^2 + OD^2 = AD^2$

Whence, $AO = 3$ m

$\therefore$ $AC = 6$ m

Now, area of the rhombus $= \frac{1}{2} 6 \times 8 = 24$ sq. m

The area to be painted, therefore, is 48 sq. m.

Cost of painting $=$ Rs $48 \times \left(\frac{7}{2}\right) =$ Rs 168 00

Exercise 24.1

1. How many carpets, 3 m by 2 m, are required to cover a hall 30 m by 12 m ?

2. If the side of a square is doubled, find the ratio of the area of the resulting square to that of the given square.

3. In exchange for a square plot of land, one of whose sides is 84 m, a man wants to buy a rectangular plot 144 m long and of the same area as the square plot. Determine the width of the rectangular plot.

4. The area of a rhombus is 98 sq. cm. If one of its diagonals is 14 cm, what is the length of the other diagonal ?

5. The cross-section of a canal is a trapezium in shape. If the canal is 10 m wide at the top, 6 m wide at the bottom and the area of the cross-section is 72 sq. m, determine its depth

6. A triangle has sides 5 cm, 12 cm and 13 cm. Find the length, to one decimal place, of the perpendicular from the vertex to the side whose length is 13 cm.

7. Determine the area of a triangle whose sides are 11 cm, 13 cm and 16 cm.

8. One angle of a rhombus is 60° and one of its sides is 4 cm. Find the length of each diagonal.

9. The area of a parallelogram is 72 sq. cm. If its altitude is twice the corresponding base, determine the base and the altitude.

10. The perimeter of a right triangle is 60 cm. Its hypotenuse is 26 cm. Find the other two sides and the area of the triangle.

11. A square and a rectangle, each have a perimeter of 48 m. If the difference between the areas of the two figures is 4 sq. m, what are the dimensions of the rectangle ?

24 2. Area of a Regular* Polygon

Let us consider a regular polygon $ABCDE$...with n sides and let us denote its centre by

*A **regular polygon** is a polygon with equal sides and angles. A **regular quadrilateral** is a square and a regular triangle is an equilateral triangle. Each angle of a regular n-gon (polygon with n sides) is equal to $\left[\frac{180\,(n-2)}{n}\right]^\circ$

A regular polygon has a point O inside it, called its **centre**, which is equidistant from all its vertices. The centre is also equidistant from all its sides. The centres of the inscribed as well as circumscribed circles will lie at the centre of the polygon.

A polygon with 5 sides is a **pentagon** ; with 6 sides, a **hexagon** ; with 7 sides, a **septagon** ; with 8 sides, an **octagon** ; with 9 sides, a **nonagon** and with 10 sides, a **decagon.**

by O. (See Fig 24 6) OA, OB, OC,... are called its **radii.**

Let us denote by h, the distance of the centre from a side, say, AB If we join O to each vertex, we divide the polygon into as many congruent triangles as its number of sides.

Now, area of $\triangle OAB = \frac{1}{2}(h)(AB)$

$\therefore$ Area of the regular polygon

$$= n\left(\frac{1}{2}\right)(h)(AB)$$

$= \frac{1}{2}(h)(p)$, where p is the perimeter of the regular polygon

i e., $\mathbf{A = \frac{1}{2}(h)(p)}$

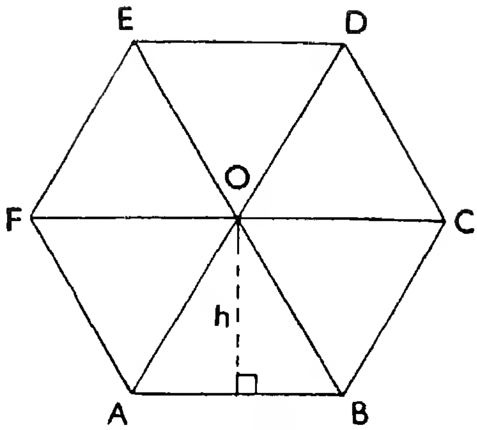


Fig. 24.6

24.3. Area of an Irregular Polygon

For irregular polygons, the area is found by decomposing the polygon into triangular and quadrilateral regions. We give below illustrations of two such decompositions. (See Fig. 24.7)

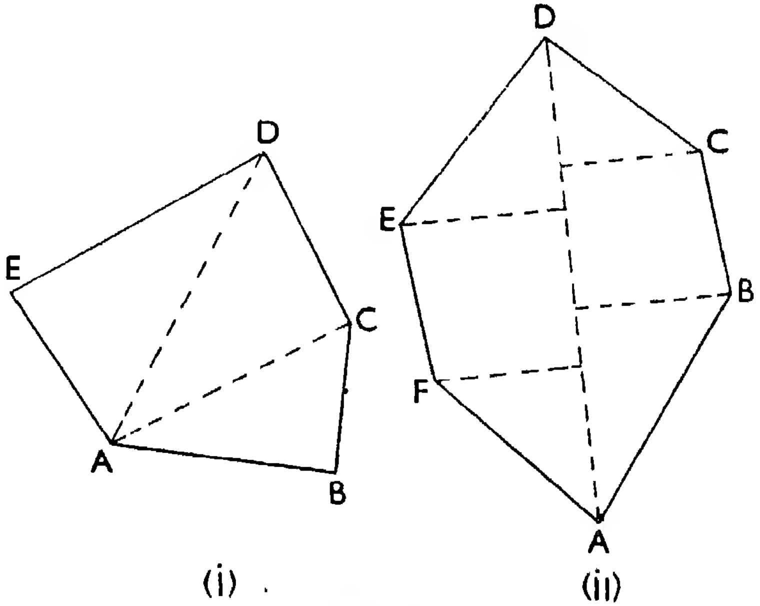


Fig. 24.7

In (*i*), the polygon *ABCDE* has been decomposed into triangular regions by drawing all possible diagonals from one of its vertices, say, *A*. We, of course, know how to find the area of a triangle.

In (*ii*), in the polygon *ABCDEF*, the longest diagonal *AD* is first drawn. (Why ?) We then draw perpendiculars from each of its remaining vertices on *AD*. The polygon is thus decomposed into right triangles and trapeziums. We, of course, know how to find the areas of triangles as also of trapeziums.

The decompositions are, obviously, not unique. It might be of interest to know that the decomposition-method is usually employed in field surveys to measure large tracts of land.

Example 1 : In a pentagonal field *ABCDE*, *AD* is its longest diagonal. (See Fig. 24.8)

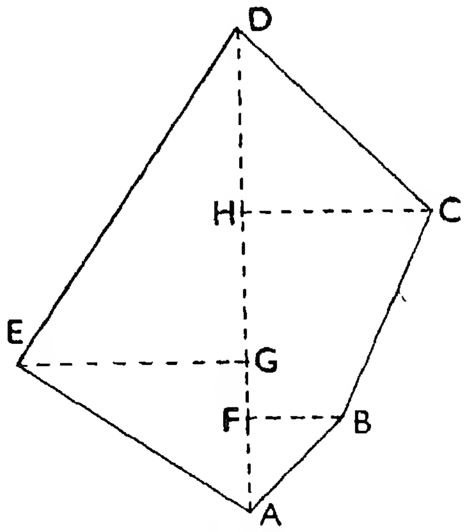


Fig. 24.8

A land-surveyor records the various measurements in his Field Book in the following form :

Metres
To *D*
130
To *E* 60—80—To *C* 50
40
25—To *B* 25
From *A*

(The Field Book is read from the bottom upwards.) Determine the area of the field.

Solution : Reading from the bottom upwards, we have

$AF = 25$ m, $FB = 25$ m, $AG = 40$ m, $GE = 60$ m,
$AH = 80$ m, $HC = 50$ m and $AD = 130$ m

$$\text{Thus, area of } \triangle DGE = \frac{1}{2}(GD)(GE)$$

$$= \frac{1}{2}(90)(60) \qquad (GD = AD - AG)$$

$$= 2700 \text{ sq. m}$$

$$\text{Area of } \triangle AGE = \frac{1}{2}(AG)(GE)$$

$$= 1200 \text{ sq. m}$$

$$\text{Area of } \triangle AFB = \frac{1}{2}(AF)(FB)$$

$$= 312.50 \text{ sq. m}$$

$$\text{Area of trapezium } FBCH = \frac{1}{2}(FB+HC)(FH)$$

$$= \frac{1}{2}(75)(55) \qquad (FH = AH - AF)$$

$$= 2062.50 \text{ sq. m}$$

$$\text{Area of } \triangle DHC = \frac{1}{2}(DH)(HC)$$

$$= 1250 \text{ sq. m} \qquad (DH = 50 \text{ m})$$

Thus, the area of the pentagonal field is 7525 sq. m or 0.7525 hectares.

Exercise 24.2

1. Find the area of a pentagonal field, given the following data in a land-surveyor's Field Book :

	Metres	
	To C	
	150	
To D 20	—120	
	80	—To B 50
To E 30	—50	
	From A	

2. A regular hexagon has a side b metres. Determine its area.

3. Find the area of a field, given the following entries in the surveyor's Field Book :

Metres
To P
120
To Q 18—100
62—To N 22
To R 27—48
34—To M 15
From L

24.4. Circle, its Circumference and Area

The perimeter of a circle is called its *circumference*. How do we find the perimeter of a circle ? We take, for instance, a circular disc, roll it, without sliding, along a straight line and measure the distance covered during one complete revolution.

The reader is advised to take several circular discs and verify that the circumference bears a constant ratio to the diameter in each case. This constant ratio is denoted by the Greek letter π (pronounced 'pi'). We thus, write

$$\frac{\textbf{Circumference}}{\textbf{Diameter}} = \pi$$

A four-place approximation to π was first given by the Hindu mathematician, Aryabhata, when he found that the circumference of a circle whose diameter is 20,000 units is 62,832 units. To four decimal places, $\pi = 3.1416$

In most calculations, the value of π used is $\frac{22}{7}$ or 3.14. **The reader must, however, note that π is not a rational number and that $\frac{22}{7}$ is only an approximation to it.**

If we denote the diameter of a circle by 'd' and its radius by 'r' we write

Circumference $(C) = \pi d$

Since $d = 2r$, it follows

$$C = \pi d = 2\pi r$$

Now how about the area of a circle whose radius is, say, r ?

We divide the circular region into n equal sectors OAB, OBC, etc. (See Fig. 24.9) If n is large, each sector is practically a triangle with height r (Why ?) and base $\frac{2\pi r}{n}$ (Why ?)

The area of one of the sectors is, therefore,

$$\frac{1}{2} \cdot \frac{2\pi r}{n} \cdot r \quad \text{or} \quad \frac{\pi r^2}{n}$$

Consequently, the area of the circle is

$$n\,\frac{(\pi r^2)}{n} \quad \text{or} \quad \pi r^2$$

i.e., $\mathbf{A = \pi r^2}$

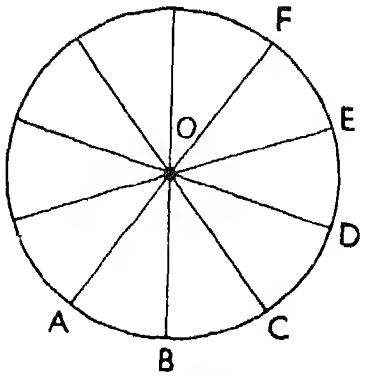


Fig. 24.9

Next, let us see how we find the area of a sector of a circle. (See Fig. 24.10)

If a sector AOB is such that $\angle AOB = \theta°$, it is obvious that **the length of the minor arc AB is**

$$\left(\frac{\theta}{360}\right) \times 2\pi r \quad \text{or} \quad \frac{\pi r\theta}{180}$$

And, area of the sector AOB is

$$\frac{\theta}{360} \times \pi r^2 \quad \text{or} \quad \frac{\pi r^2\theta}{360}$$

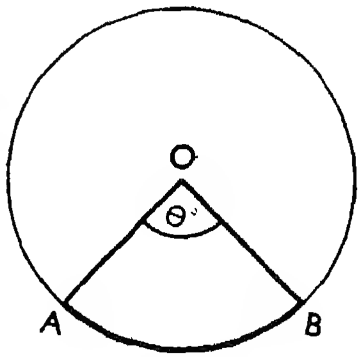


Fig. 24.10

Example 1 : The inner circumference of a circular race track is 440 m. The track is everywhere 14 m wide. Calculate the cost of

(*i*) levelling the track at the rate of 20 paise per sq. m.

(*ii*) putting up a fence along the outer circle at the rate of Rs 2 00 per metre.

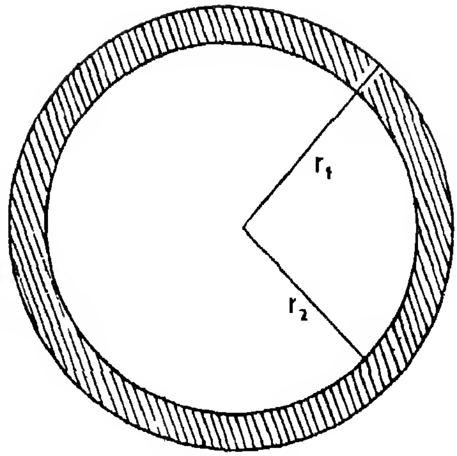


Fig. 24.11

Solution : (*i*) Let us denote by r_1 and r_2, respectively, the radii of the outer and inner circles. (See Fig. 24.11)

Since the circumference of the inner circle is 440 m, we have

$$2\pi r_2 = 440$$

Whence $\quad r_2 = 70 \text{ m} \left(\text{Since} \quad \pi = \frac{22}{7}\right)$

Thus, radius of the outer circle is 84 m. (Why?)

$\therefore$ Area of the track $= \pi r_1^2 - \pi r_2^2$

$= 6776$ sq. m

At 20 paise per sq. m, the cost of levelling the track is Rs 1355.20

(*ii*) The circumference of the outer circle $= 2\pi r_1$

$= 528$ m

At Rs 2.00 per metre, the cost of putting up a fence is Rs 1056.00

Example 2 : A circle of radius 3.5 m is divided into three sectors with central angles 30°, 120° and 210°. Find the area of each sector.

Solution : The area of the circle $= \pi r^2 = 38.5$ sq. m

(*i*) Thus, the area of the sector with central angle 30°

$$= 38.5 \times \frac{30}{360}$$

$= 3.21$ sq. m

(*ii*) The area of the sector with central angle 120°

$$= 38.5 \times \frac{120}{360}$$

$= 12.83$ sq. m

(*iii*) And, the area of the sector with central angle 210°

$$= 38.5 \times \frac{210}{360}$$

$= 22.47$ sq. m

Exercise 24.3

1. Four circular windows, each of radius 28 cm, are to be fitted with glass. Calculate the cost of glass at the rate of Rs 4.25 per sq. m.
2. The length of a minute-hand on a wall clock is 10.5 cm. Find the area swept by the minute-hand in 10 minutes.
3. A wire is in the form of a circle of radius 42 cm. Determine the side of the square into which it can be bent.
4. The diameter of a wheel is 1.26 m. How far will it travel in 500 revolutions ?
5. Given a circle with radius 3.6 cm. Find the perimeter and the area of its sector with central angle 36°.
6. A park is in the form of a square, one of whose sides is 100 m. At the centre of the park is a circular lawn. If the area of the park excluding the lawn is 8614 sq. m , find the cost of raising the lawn at the rate of Re 1 00 per sq. m.

24.5. Cuboids and Cubes

A chalk-box, a match-box are examples of **cuboids.** A cuboid has 6 plane surfaces, called **faces.** (See Fig. 24.12) Each face is a rectangle.

We note that corrresponding to any face, there is only one other face which does not intersect it. In fact, the other face is parallel to it. A **cuboid, therefore, has three pairs of parallel faces and faces in each pair are congruent.**

Further, any two adjacent faces meet along a line-segment, called an **edge** of the cuboid **A cuboid has 12 edges. Also, a cuboid has 8 corners, called its vertices.** We note that at each vertex, three edges and three faces meet. It follows that the pairs of opposite edges are equal. This need not be so in case of adjacent edges.

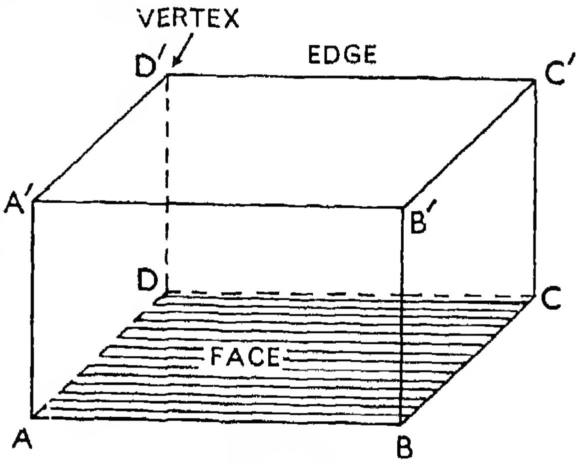


Fig. 24.12

In other words,

(i) $AB = A'B' = CD = C'D'$

(*ii*) $BC = B'C' = AD = A'D'$
(*iii*) $BB' = AA' = CC' = DD'$

The twelve edges of a cuboid, therefore, can have only three sets of four equa measurements. It is customary to call the first set, (*i*) above, as the length and denote it by 'l' ; the second, (*ii*) above, as breadth (width) and denote it by 'b' ; and the third, (*iii*) above, as the height (depth) and denote it by 'h'.

A. Surface Area

It is obvious, that the total surface area of the cuboid is the sum of the areas of its six faces, i.e.,

Total Surface Area of the Cuboid = 2(lb+bh+hl)

When all the edges of a cuboid are equal, we call it a **cube.** Each face of a cube is, therefore, a square. If l is the length of an edge (side) of a cube, we have

Total Surface Area of the Cube = 6 l²

B. Volume and its Measure

By the **volume** of a solid (3-dimensional figure) is meant the extent of the 3-space (region of the space) enclosed by it. The unit of measurement of volume is a cube. If, for instance, the unit is a cube with side 1 cm, the unit of volume is 1 cu. cm. If, on the other hand, the unit is a cube with side 1 m, the unit of volume is 1 cu. m. **The volume of a solid, therefore, is the number of unit-cubes contained by the solid.**

Now, how about the volume of a cuboid, say, 6 cm long, 3 cm wide and 4 cm high ?

Let us use a cube of side 1 cm as the unit-cube. It is obvious, that 4 layers of 18 (or, 6×3) unit-cubes are required to fill the 3-space contained by the cuboid. (See Fig. 24.13)

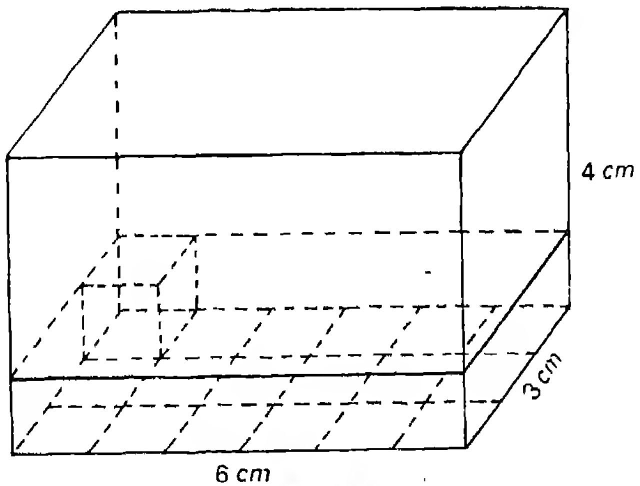


Fig. 24.13

In other words, the cuboid will contain 72 unit-cubes, i.e., the volume of the cuboid is 72 (or $6\times3\times4$) cu. cm.

The formula for the **volume V of a cuboid with dimensions l, b and h can, therefore, be stated as**

$$\mathbf{V = l\times b\times h}$$

It follows that the volume of a cube with a side l unit is

$$\mathbf{V = l^3}$$

Example 1 : Find the depth of a tank whose base is a square of side 5 m, if it can hold as much water as another tank whose dimensions are 10 m by 5 m by 4 m.

Solution : Let the depth of the tank be h metre.

Then, $25\,h = 200$

or, $h = 8$

Thus, the depth of the tank is 8 m.

Example 2 : The outer measurements of a closed wooden box are 42 cm $\times$ 30 cm $\times$ 27 cm. If the box is made of wood 1 cm thick, determine the capacity of the box.

Solution : Since the wood is 1 cm thick, the inner measurements of the box are :

length = 42—2 = 40 cm

width = 28 cm

height = 25 cm

$\therefore$ The capacity of the box = inner volume of box

= 28,000 cu. cm

Exercise 24.4

1. The dimensions of a cinema hall are 100 m, 50 m and 18 m. How many persons can sit in the hall, if each requires 150 cu. m of air ?
2. The dimensions of a field are 15 m by 12 m. A pit, 8 m long, 2.5 m wide and 2 m deep is dug in one corner of the field and the earth removed has been evenly spread over the remaining area of the field. Calculate, by how much is the level of the field raised?
3. What is the area of the card-board needed to make a rectangular box 12 cm long, 8 cm wide and 6 cm high ?
4. Two cubes, each with 12 cm edge, are joined end to end. Find the surface area of the resulting cuboid.
5. A classroom is 7 m long, 6.5 m wide and 4.5 m high. It has one door 3 m by 1.4 m and four windows, each 2m by 1 m. Find the area of the walls that has to be whitewashed (inside only).
6. A tank 15 m long, 10 m wide, and 6 m deep is to be made. It is open at the top. Determine the cost of iron-sheet, at the rate of Rs 2.50 per metre, if the sheet is 4 m wide.

7. The areas of three adjacent faces of a cuboid are x, y and z. If its volume is V, prove that $V^2 = xyz$.

24.6. Right Prisms

Let us consider two triangles, of the same size and shape, lying in parallel planes. Let us place them in such a way that when their corresponding vertices are joined by segments, three rectangles are formed. (See Fig. 24.14) The resulting figure is called a **triangular right prism.** We note that a triangular right prism has 6 **vertices**, 9 **edges** and 5 **faces**. Of these 5 faces, 3 are rectangular and the other two (which are parallel to each other) are triangular.

Similarly, by placing, say, two pentagons of the same size and shape in parallel planes and joining their corresponding vertices by segments, we have a figure called **pentagonal right prism.** (See Fig. 24.15) We note that there are 5 rectangular faces and 2 pentagonal faces (which are parallel to each other).

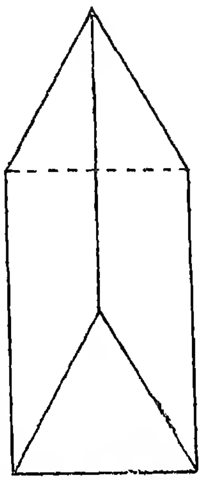


Fig. 24.14

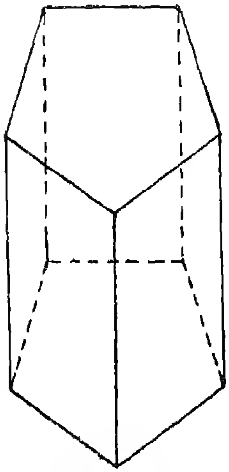


Fig. 24.15

In general, a **right prism** has two parallel faces in the form of a polygon of any number of sides often called its **bases** The distances between the bases is called the **height** of the right prism. The remaining faces called **lateral faces** are, however, all rectangles. The number of these rectangular faces is equal to the number of the sides of the base. If any of the faces (other than the two bases) is not a rectangle (rather, it is a parallelogram), then the figure is no longer a right prism ; it is simply called a **prism.**

Note 1 : A right prism need not necessarily rest on one of its bases. (See Fig. 24.16) The prism does not rest on either of the two triangular bases, ABC or $A'B'C'$. In such cases, it would be better to refer the bases as **ends** and height as **length.**

Note 2 : The reader should note that a **cuboid** (as also a cube) is a **right prism.** (Why ?) What are its bases ? A cuboid is also called a **rectangular right prism.**

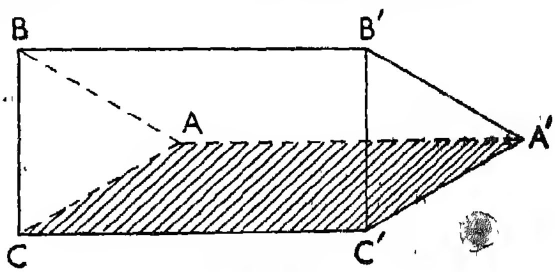


Fig. 24.16

A. Surface Area

Since the faces of a right prism are polygons, we can easily compute its surface area A. It is obvious that

$$\mathbf{A = 2\,(Area\ of\ one\ end) + (Area\ of\ the\ lateral\ faces)}$$

The reader should note that the

$$\mathbf{Area\ of\ the\ lateral\ faces = (Perimeter\ of\ the\ base) \times (Height)}$$

Area of the lateral faces is called the **area of the lateral surface** or, briefly, the **lateral surface** and the (total) surface area is called the **area of the whole surface** or, briefly, the **whole surface** of the right prism.

B. Volume

We have already learnt (Section 24.5 B) that the volume V of a cuboid (rectangular right prism) is given by

$$V = l \times b \times h$$

i.e.,

$$\mathbf{V = (Area\ of\ the\ base) \times (Height)}$$

Since the volume has been defined as the number of unit-cubes contained by the solid, the formula above should also give us the volume of any right prism, irrespective of the shape of its base. (The proof of this formula is beyond the scope of this book) For instance, if the end of a right prism is a triangle, with one of its sides 8 cm and the corresponding altitude 6 cm and the length of the prism is 25 cm (See Fig. 24.17), its volume V is

$$V = \left(\frac{1}{2} \times 8 \times 6\right) \times 25$$

or,

$$V = 600 \text{ cu. cm}$$

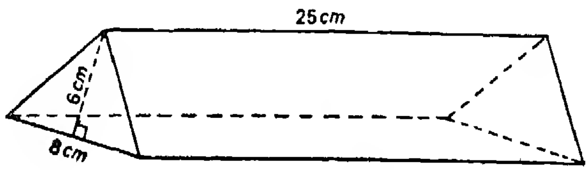


Fig. 24.17

Example 1 : The base of a right prism is in the form of a triangle ABC right-angled at C. If $AC = 5$ cm, $AB = 13$ cm and the height of the prism is 24 cm, find the whole surface and the volume of the prism.

Solution : Since ABC is right-angled at C, we have

$$BC = 12 \text{ cm}$$

Now, the surface area of the prism

$$= 2\ (\text{Area of the base}) + (\text{Area of the lateral faces})$$

$$= 2\times\left(\frac{1}{2}\times 12\times 5\right)+(12+5+13)\times 24$$

$$= 780 \text{ sq. cm}$$

And, the volume of the prism

$$= (\text{Area of the base})\times(\text{Height})$$

$$= 720 \text{ cu. cm}$$

Therefore, the whole surface of the prism is 780 sq. cm and its volume is 720 cu. cm.

Exercise 24·5

1. The base of a right prism is an equilateral triangle, with a side 2·8 cm. If the height of the prism is 15 cm, find its lateral surface and its volume.

2. Find the cost of polishing the lateral surface of a granite column, 5 m high and standing on an octagonal base with each side 20 cm, at the rate of Rs 2.50 per sq. m.

3. The base of a right prism is a trapezium whose parallel sides are 10 cm and 8 cm respectively, and the distance between the parallel sides is 9 cm. If the height of the prism is 15 cm, determine its volume.

4. A tent is in the shape of a triangular prism 2 5 m long. The measurements of the ends are given in Fig. 24.18.

   If canvas is used for its ends as well as its sides but not for its floor, determine the cost of canvas at the rate of Rs 4.50 per sq. m. How much is the air-content of the tent ?

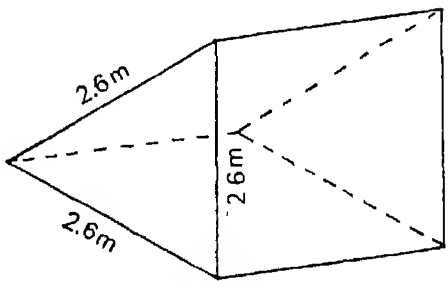
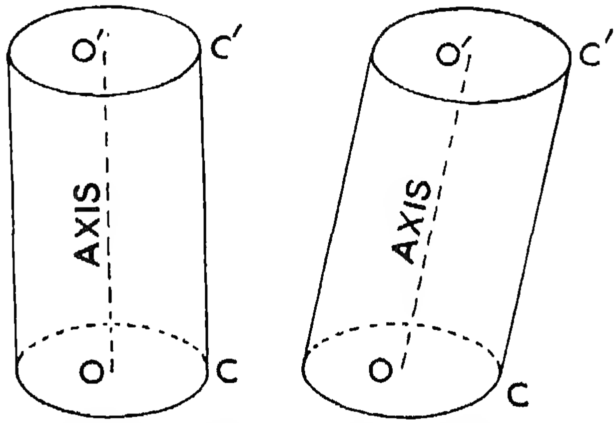


Fig. 24.18

5. Water flows in a canal at 0·5 m per second. The cross-section of the canal, from its bottom to the water level, is a trapezium 2.5 m at the bottom and 4 m at the top. If the water is 1.5 m deep, how much water does pass through a particular cross-section of the canal in a minute ?

24.7. Right Circular Cylinders

The reader may recall that the lateral faces of a right prism are rectangles and its base is in the form of a polygon. In daily life, however, we come across several objects that have no rectangular faces ; for instance, tin cans, pipes, etc. These objects have curved (lateral) surface. What is the shape of their closed ends ? Certainly, in most cases, they are flat and have no straight edges and are in the form of a circle*. These objects are examples of solid figures called **circular cylinders**. If, in addition, the line joining the centres of the circular bases is perpendicular to the base, the solid figure is called the **right circular cylinder**. (See Fig. 24.19)



RIGHT CIRCULAR CYLINDER (i) CIRCULAR CYLINDER (ii)

Fig. 24.19

The line joining the centres is called the **axis** of the cylinder.

A. Surface Area

It is obvious that the surface area of a right circular cylinder would be the area of its bases added to the area of its curved surface. The area of the bases is easy to compute since each base is a circle. How about the area of the curved surface ? As in the case of a right prism, the area of the curved surface of a right circular cylinder is obtained by multiplying the perimeter of its base by its height.

---

*It is possible that the closed end may have no straight edge and also may not be in the form of a circle. For instance, it may be in the form of an ellipse. In that case, the solid figure would be called an **elliptic cylinder**. However, this discussion is beyond the scope of this book.

If 'r' is the radius of the base and 'h' the height of the right circular cylinder, its

$$\textbf{Curved Surface} = 2\pi rh$$

and

$$\textbf{Whole Surface} = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$$

B. Volume

As in the case of a right prism, the volume of a right circular cylinder is obtained by multiplying the area of its base by its height. Thus, the volume V of the right circular cylinder is

$$V = \pi r^2 h$$

where r and h are respectively the radius of the base and the height of the cylinder.

Example 1 : A cylindrical tank has a capacity of 6160 cu. m. Find its depth, if the diameter of its base is 28 m. Also, calculate the cost of painting its inside curved surface at the rate of Rs 2 80 per sq. m.

Solution : If we denote the depth of the tank by h, we have

$$\pi(14)^2 h = 6160$$

Whence, $$h = 10$$

Thus, the tank is 10 m deep.

The area of the curved surface is, therefore, $2\pi rh$ or 880 sq. m.

At Rs 2 80 per sq. m, the cost of painting the inside curved surface is Rs 2464.00.

Exercise 24.6

1. A cylindrical vessel, without a lid, has to be tin-coated on both its sides. If the radius of its base is $\frac{1}{2}$ m and its height is 1·4 m, calculate the cost of tin-coating at the rate of Rs 2.25 per 1000 sq. cm. (Use $\pi=3{\cdot}14$)
2. 10 cylindrical pillars of a building have to be cleaned. If the diameter of each pillar is 50 cm and the height 4 m, what will be the cost of cleaning these at the rate of 50 paise per sq. m ? (Use $\pi = 3\ 14$)
3. The inner radius of a pipe is 2.5 cm. How much water will 10 m of this pipe hold ? (Use $\pi = 3.14$)
4. A powder tin has a square base with side 8 cm and height 13 cm. Another is cylindrical with the radius of its base 7 cm and its height 15 cm. Which of the two contains more powder ? How much is the difference in their capacities ?
5. A rectangular piece of paper is 22 cm long and 10 cm wide. A cylinder is formed by rolling the paper along its length. Find the volume of the cylinder.

6. A 20 m deep well with diameter 7 m is dug up and the earth from digging is spread evenly to form a platform 22 m × 14 m. Determine the height of the platform.

7. The diameter of a roller 120 cm long is 84 cm. If it takes 500 complete revolutions to level a playground, determine the cost of levelling at the rate of 30 paise per sq m.

24.8. Right Pyramids

The reader has perhaps heard of the pyramids of Egypt. A pyramid is a solid figure with its base as a polygon and its lateral faces as triangles. (See Fig. 24 20)

The triangles meet at a common point called the **vertex** and the length of the perpendicular segment from the vertex to its base is called the **height** of the pyramid.

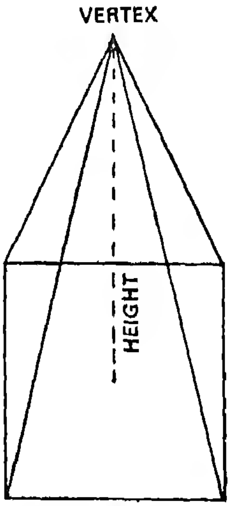


Fig. 24.20

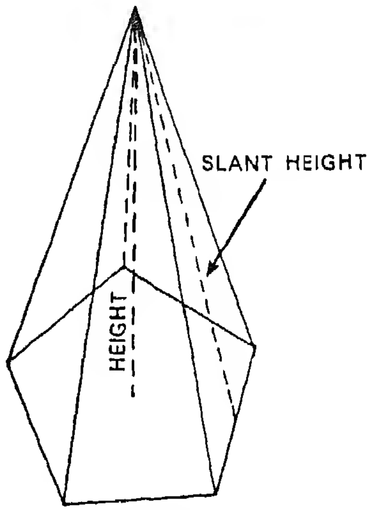


Fig. 24.21

If the base is a regular polygon and the foot of the perpendicular segment from the vertex coincides with the centre of the polygon, the solid figure is called a **right pyramid.** (See Fig. 24.21)

The length of the perpendicular segment from the vertex to any of the sides of the regular polygon (which form the base of the right pyramid) is called the **slant height** of the right pyramid. We next find its lateral surface and volume.

Now, the area of a triangular face $= \frac{1}{2}$ (Side of the polygon) × (Slant height)

Thus, **the lateral surface** $= \frac{n}{2}$ **(Side of the polygon) × (Slant height)**

where n is the number of the sides of the polygon.

In other words

$$\textbf{The lateral surface} = \frac{1}{2}\ \textbf{(Perimeter of the base)} \times \textbf{(Slant height)}$$

It can easily be verified that the volume V of the right pyramid is one-third the volume of the right prism with the same base and the same height

i.e., $$V = \frac{1}{3}\ \textbf{(Area of the base)} \times \textbf{(Height)}$$

24.9. Right Cones

The reader may recall that the lateral faces of a right pyramid are triangles and its base is a regular polygon. In daily life, however, we come across several objects, such as ice-cream cones, joker's cap, etc. These objects have curved (lateral) surface and circular base They are examples of solid figures called **circular cones** If, in addition, the perpendicular from the vertex V to its base passes through the centre C of the base, the solid figure is called a **right circular cone** and VC its **height.** (See Fig 24.22) The length of the segment joining the vertex to any point on the circle is called the **slant height** of the right circular cone We next find the curved surface and the volume of the right circular cone.

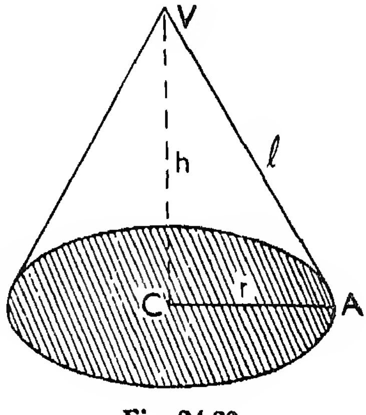


Fig. 24.22

As, in the case of a right pyramid, the area of the curved surface of a right circular cone is obtained by multiplying half the perimeter of its base by its slant height.

If 'r' is the radius of the base and 'l' the slant height of a right circular cone, the **area of its curved surface,** A, is

$$A = \frac{1}{2}(2\pi r)l = \pi rl$$

The **total surface area, S, is,** therefore,

$$S = \pi r^2 + \pi rl = \pi r(r+l)$$

Similarly, the **volume, V,** of a right circular cone whose height is 'h' and radius of the base is 'r', is

$$V = \frac{1}{3}\ \textbf{(Area of the base)} \times \textbf{(Height)}$$

or, $$V = \frac{1}{3}\pi r^2 h$$

Example 1 : A right pyramid in Egypt is 150 m high and has a square base with a side 220 m. Determine its volume and lateral surface.

Solution : The volume, $V = \frac{1}{3}$(Area of the base) $\times$ (Height)

$$= \frac{1}{3}(220)^2(150)$$

$$= 2420000 \text{ cu. m}$$

From Fig. 24·23, it is obvious that the slant height, l, is

$$l = \sqrt{(110)^2+(150)^2}$$
$$= 186 \text{ (approx.)}$$

$\therefore$ Lateral Surface $= \frac{1}{2}$ (Perimeter of the base) $\times$ (Slant height)

$$= \frac{1}{2}(880)(186)$$
$$= 81840 \text{ sq. m}$$

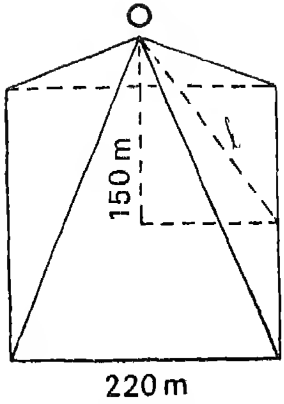


Fig. 24.23

Thus, the volume of the pyramid is 2420000 cu. m and its lateral surface is 81840 sq. m.

Example 2 : The circumference of the base of a 10 m high conical tent is 44 m. Calculate the area of the canvas used in making the tent. Also, find the volume of the air contained in it.

Solution : Since the circumference $C = 2\pi r = 44$, it is easy to calculate the value of r, namely, $r = 7$.

The slant height, l, therefore is

$$l = \sqrt{(7)^2+(10)^2}$$
$$= 12.2 \text{ (approx.)}$$

Thus, curved surface of the tent $= \pi rl = 268.4$

And, the volume, V, of the air contained in the tent is

$$V = \frac{1}{3}\pi r^2 h = 513.3$$

Thus, the canvas required is 268.4 sq. m and the volume of the air contained is 513.3 cu. m.

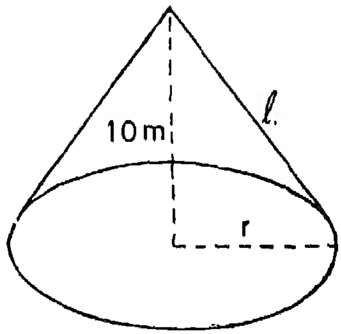


Fig. 24.24

Example 3 : If the radii of the ends of a bucket 45 cm high are 28 cm and 7 cm (See Fig. 24.25), determine its capacity and the surface area.

Solution : The bucket can be viewed as a difference of two right circular cones OAB and OCD.

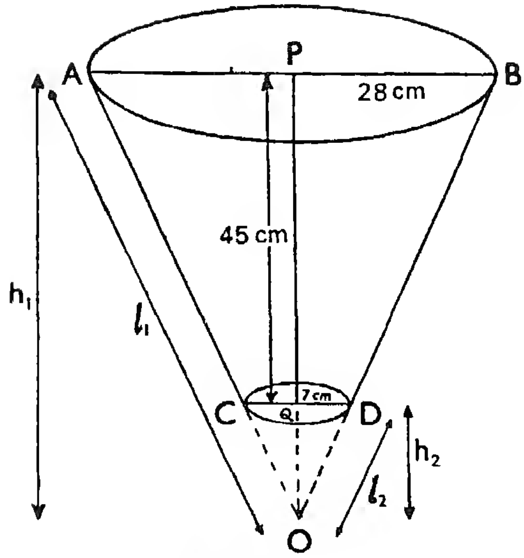


Fig. 24.25

We first need to determine the respective heights h_1 and h_2 of the two cones.

$\triangle$s OPB and OQD are similar. (Why ?)

$$\therefore \quad \frac{h_1}{h_2}=\frac{28}{7} \qquad (i)$$

But $h_1 = 45+h_2$. Substituting this value in (*i*), we obtain $h_2 = 15$; whence, $h_1 = 60$.

Now, the volume of the bucket $=$ Volume of the cone OAB—Volume of the cone OCD

$$= \frac{1}{3}\pi\,(28)^2(60)-\frac{1}{3}\pi(7)^2(15)$$

$$= 48510 \text{ cu. cm}$$

We also need to determine the respective slant heights l_1 and l_2 of the cones OAB and OCD.

$$l_1 = \sqrt{(28)^2+(60)^2} = 66.2 \text{ (approx.)}$$

and $$l_2 = \sqrt{(7)^2+(15)^2} = 16.6 \text{ (approx.)}$$

Thus, the surface area of the bucket

$$= \pi r_1 l_1-\pi r_2 l_2+\pi r_2^2 \qquad \text{(Why ?)}$$

$$= 5614.4$$

The capacity of the bucket, therefore, is 48510 cu. cm and its surface area is 5614.4 sq. cm.

Exercise 24.7

1. The base of a right pyramid is a square with side 10 cm. If the height of the pyramid is 12 cm, calculate its lateral surface and the volume
2. A tower whose base is in the form of a regular hexagon, with side $4\sqrt{3}$ m, has a roof in the form of a right pyramid, 8 m high. Determine the cost of tiling the roof at the rate of Rs 4.75 per sq. m.
3. A circus tent is cylindrical to a height 3 m and conical above it. If its diameter is 105 m and slant height of the cone is 53 m, calculate the total area of the canvas required.
4. The radius and the height of a right circular cone are in the ratio 5 : 12. If its volume is 314 cu. m, find the slant height and the radius. (Use $\pi = 3.14$)
5. A right pyramid has a square base whose area is 180 sq. m. If the slant height and the height of the pyramid are in the ratio 3 : 2, determine its volume and lateral surface.
6. The area of the base of a right circular cone is 78.5 sq cm. If its height is 12 cm, find its volume and the curved surface. (Use $\pi = 3.14$)

24.10. Spheres

Let us recall that a circle is a set of points in a plane each of whose points is at the same (constant) distance from a given point in the plane. What if we consider **points in the 3-space, each equidistant from a given point** ? The set of these points would form a surface, very much like the surface of a ball. This set of points is called a **sphere**. The given point is called the **centre** and the constant distance the **radius** of the sphere.

Now, let us consider segments with their end-points on the sphere. It is obvious that not all these segments are equal. In fact the segments that pass through the centre of the sphere are the longest and are equal to twice the radius. Such segments are called **diameters** of the sphere.

We state, without proof, that the surface area, A, and the volume, V, of a sphere whose radius is 'r' are given by the formulae

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

(The proofs of these formulae are beyond the scope of this book.)

Example 1 : The diameter of a spherical shot-put, made of brass, is 14 cm. Calculate its (*a*) surface area and (*b*) the volume of brass it contains. $\left(\text{Use } \pi = \frac{22}{7}\right)$

Solution : (*a*) The radius, r, of the shot-put = 7 cm (Why ?)

$\therefore$ The surface area of the shot-put $= 4\left(\frac{22}{7}\right)(7)^2$

$= 616$ sq. cm

(*b*) And, the volume of the shot-put $= \frac{4}{3}\left(\frac{22}{7}\right)(7)^3$

$= 1437.3$ cu. cm

Exercise 24.8

1. The spherical valve of a tank is 21 cm in diameter Calculate its surface area and volume. $\left(\text{Use } \pi = \frac{22}{7}\right)$
2. The diameter of a hemispherical* bowl is 12 cm. Calculate its internal surface area and also its capacity. (Use $\pi = 3{\cdot}14$)
3. A toy is in the form of a cone mounted on a hemisphere. The diameter of the base of the cone is 6 cm and its height is 4 cm. Calculate the surface area of the toy. (Use $\pi = 3.14$)
4. Given that the volume of a metal-sphere is 38808 cu. cm, find its radius and hence its surface area. $\left(\text{Use } \pi = \frac{22}{7}\right)$
5. A spherical stone, whose density is 2 gm/cu. cm, is 42 cm in diameter. Can a person, who can lift 100 kg, lift this stone ? (Use $\pi = 3.14$)
6. A cylindrical boiler, 2 m high, is 3.5 m in diameter. It has a hemispherical lid. Find the volume of its interior (including of the part covered by the lid). $\left(\text{Use } \pi = \frac{22}{7}\right)$
7. A cone is 8.4 cm high and the radius of its base is 2.1 cm. It is melted and recasted into a sphere. Determine the radius of the sphere.
8. How many lead balls, each of radius 1 cm, can be made from sphere, whose radius is 8 cm ?

---

*If a sphere is sliced into two equa lparts by a plane passing through its centre, each part is called a **hemisphere**.

9. A spherical shell of lead whose external diameter is 18 cm, is melted into a right circular cylinder, 8 cm high and 12 cm in diameter. Find the inner diameter of the shell.

10. The diameter of a sphere is 6 cm. It is melted and drawn into a wire of diameter 0.2 cm. Find the length of the wire.

24.11. Areas of Irregular Plane Figures

The reader is already familiar with the method of finding the areas of irregular plane figures by using graph paper. The areas thus formed are, however, only approximate. The reader is advised to draw some irregular plane figures, in particular, maps of lakes, gardens, cities, etc., and determine their areas by the above method The reader may also determine the areas of school play-grounds, laboratories, etc.

24.12 We give below a summary of all the formulae used in this chapter.

A. Plane Figures

		Area	*Perimeter*	*Remarks*
1.	Rectangle	$a\times b$	$2(a+b)$	a : length ; b : breadth
2	Triangle	$\frac{1}{2}b\times h$	$a+b+c$	a, b, c : sides , h : altitude corresponding to side 'b'
		$\sqrt{s(s-a)(s-b)(s-c)}$		$s=\frac{a+b+c}{2}$
3.	Parallelogram	$b\times h$		b : side ; h : corresponding altitude
4.	Rhombus	$\frac{1}{2}\ d_1\times d_2$		d_1, d_2: lengths of its diagonals
5.	Trapezium	$\frac{1}{2}(a+b)h$		a, b : parallel sides ; h : the distance between the parallel sides
6.	Circle	πr^2	$2\pi r$	r : radius
7.	Sector of a Circle	$\frac{\theta}{360}\pi r^2$	$\frac{\theta}{180}\ \pi r$	θ : central angle ; r : radius of the sector

B. Solid Figures

		Lateral/ Curved surface	*Whole surface*	*Volume*	*Remarks*
1.	Cuboid		$2(lb+bh+hl)$	$l\times b\times h$	l : length ; b : breadth ; h : height
2.	Cube		$6l^2$	l^3	l : side
3.	Right Prism	(Perimeter of the base) $\times$ (Height)	2 (Area of one end) + Lateral surface	(Area of the base) $\times$ Height	
4.	Right Circular Cylinder	$2\pi rh$	$2\pi r(r+h)$	πr^2h	h : height ; r : radius
5.	Right Pyramid	$\frac{1}{2}$ (Perimeter of the base) $\times$ (Slant height)	Area of the base + Lateral surface	$\frac{1}{3}$ (Area of the base) $\times$ (Height)	
6.	Right Circular Cone	πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2h$	h : height ; r : radius ; l : slant height
7.	Sphere		$4\pi r^2$	$\frac{4}{3}\pi r^3$	r : radius

APPENDICES

Factorizing a Quadratic Polynomial over R

A. To be able to factorize ax^2+bx+c, where $a, b, c \in R$ and $a \neq 0$, we need to determine two numbers r and s, such that

$$r+s = b \qquad (1)$$

And,
$$rs = ac \qquad (2)$$

Therefore,
$$(r-s)^2 = b^2-4ac$$

Whence,
$$r-s = \pm\sqrt{b^2-4ac} \qquad (3)$$

From (1) and (3), upon addition,

$$r = \frac{b \pm \sqrt{b^2-4ac}}{2}$$

Upon substitution in (1),

$$s = \frac{b \mp \sqrt{b^2-4ac}}{2}$$

B. We, therefore, split the term in x and write

$$ax^2+bx+c = ax^2+\left(\frac{b \pm \sqrt{b^2-4ac}}{2}\right)x+\left(\frac{b \mp \sqrt{b^2-4ac}}{2}\right)x+c$$

$$= ax\left[x+\left(\frac{b \pm \sqrt{b^2-4ac}}{2a}\right)\right]+\left(\frac{b \mp \sqrt{b^2-4ac}}{2}\right)\left[x+\left(\frac{2c}{(b \mp \sqrt{b^2-4ac}}\right)\right] \qquad (4)$$

In (4), we rationalize the denominator in $\dfrac{2c}{b \mp \sqrt{b^2-4ac}}$ by multiplying the numerator and denominator by $b \pm \sqrt{b^2-4ac}$ and obtain $\dfrac{2c(b \pm \sqrt{b^2-4ac})}{b^2-(b^2-4ac)}$ or $\dfrac{b \pm \sqrt{b^2-4ac}}{2a}$. Thus, (4) can be rewritten as:

$$ax^2+bx+c = ax\left[x+\left(\frac{b \pm \sqrt{b^2-4ac}}{2a}\right)\right]+\left(\frac{b \mp \sqrt{b^2-4ac}}{2}\right)\left[x+\left(\frac{\pm\sqrt{b^2-4ac}}{2a}\right)\right]$$

$$= \left[x+\left(\frac{b \pm \sqrt{b^2-4ac}}{2a}\right)\right]\left[ax+\left(\frac{b \mp \sqrt{b^2-4ac}}{2}\right)\right]$$

$$= a\left[x+\left(\frac{b \pm \sqrt{b^2-4ac}}{2a}\right)\right]\left[x+\left(\frac{b \mp \sqrt{b^2-4ac}}{2a}\right)\right]$$

APPENDIX VII

Theorem : **If a and b are any two real numbers such that ab = 0, then either a = 0 or b=0 or both a = 0 and b = 0.**

Given : $ab = 0$

To prove : Either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$.

Proof : If $ab = 0$ and if $a = 0$, the theorem needs no proof.
If $ab = 0$ and if $b = 0$, again, the theorem needs no proof
If $ab = 0$ and if $a \neq 0$, **let us prove** that $b = 0$

Since $a \neq 0$, $\frac{1}{a}$ is defined.

Consider $ab = 0$ (1)

Multiplying both sides of (1) by $\frac{1}{a}$, we obtain :

$$\frac{1}{a}(ab) = \frac{1}{a}(0)$$

i.e., $\frac{1}{a}(ab) = 0$ (Why ?)

or, $\left(\frac{1}{a} \cdot a\right)b = 0$ (Associativity of multiplication)

i.e., $1 \cdot b = 0$ (Why ?)

or, $b = 0$ (Why ?)

Thus if $ab = 0$ and $a \neq 0$, then $b = 0$

Similarly, we can show that if $ab = 0$ and $b \neq 0$, then $a = 0$

This proves the theorem.

Q.E.D.

APPENDIX VIII

Rates and Taxes

Income Tax (Salaried People)

The Government performs many functions which are necessary for the national security and the welfare of its people. It, therefore, levies taxes on the incomes of the individuals and businesses

For salaried people, income tax is calculated for each financial year, April 1—March 31. Their income tax is deducted, at source, by the employer. The employer forwards the money, thus collected, to the treasury. The individual is still required to file income tax return, every year, unless he/she is otherwise exempted from filing.

Neither it is possible, nor it is intended, to give, in these few pages, a complete set of instructions for filing a tax return. The tax laws and additions, deletions and amendments to these are numerous. We give below a proforma for computation of income tax for 1975-1976 :

(A) (i) Gross emoluments received during the financial year.

[This includes total earnings (including allowances) from salaries and other sources such as honorariums, etc]

(ii) Less amount deposited under the additional emoluments scheme (CD Act) 1974

(iii) Less admissible deduction on account of House Rent Allowance

Balance (A)

(B) Less standard deduction in respect of expenditure incidental to employment.

[Standard deduction is computed at the rate of 20% up to Rs 10,000/- of the Balance in (A) plus 10% of the remaining amount, subject to a maximum of Rs 3,500/-]

Balance (B)

(C) Less expenditure on higher education of dependent children.

[The deduction allowed is Rs 1000/- if the dependent child (children) is (are) studying for a degree or Post-Graduate Diploma in Medicine, Architecture, Engineering or Technology. For other degrees or Post-Graduate Diplomas the deduction is limited to Rs 500/-]

Balance (C)

(D) Rebate on account of contributions to General Provident Fund (G.P.F.), Premium on Life Insurance (P.L.I.), 10 years cumulative time deposits.

[The total amount of rebate is calculated as 100% of the first Rs 4000/- of the amount at (D), plus 50% of the balance, subject to the condition that total deposits at (D) should not exceed 30% of the total income.]

Balance (D)

(E) Taxable Income

[Balance at 'D' rounded off to the nearest Rs 10/-]

(F) Gross Tax

[This is computed from a tax rate schedule provided by the Department of Central Revenues, Govt. of India]

(G) Surcharge

[10% of gross tax in (F)]

(H) Total Tax (F+G)

II. Postal Information

	Rs P
Postcard	
Single	0.15
Reply	0.30
Inland letter	0.20
Letters	
For the first 15 g	0.25
Every additional 15 g or part	0.15
Printed Materials, Books, etc.	
Not exceeding 100 g	0.15
Every additional 50 g or fraction thereof	0.05
Parcels (Ordinary)	
Inland	
Every 500 g or fraction thereof	1.50
Every additional 500 g or fraction thereof	1.50
Registration fee per article	2 00
Acknowledgement Due (AD)	0.25
Parcels (Air)	
Inland	
Surcharge for conveyance by Air (payable in addition to ordinary postage)	
Parcels each 100 g or part	0-30
Every additional 100 g or part	0.30
Insurance Fee	
Value up to Rs 100/-	1 00
Every additional Rs 100/- or fraction thereof (up to Rs 5000/-)	0.50
Every additionel Rs 1000/- or fraction thereof (above Rs 5000/-)	3.00
Money Order	
Fee for every Rs 10/- up to Rs 20/-	0.25
For every additional Rs 20/- or fraction thereof	0.50

Post Cards	**By Surface Mail** Rs P	**By Air Mail** Rs P	**Aerograms** Rs P
For Afganistan, Sri Lanka and Burma	0.55	1.20	1.40
For A.O.P.U.* countries	0.80	1.40	1.60
For all other countries	1.10	1.40	1.60

Telegrams	Rs P
Inland ordinary : minimum 8 chargeable words	2.00
Every additional word	0.25
Inland Express : minimum 8 words	4.00

*These include Australia, Japan, Laos, Thailand, New Zealand, Indonesia, Korea, Philippines Republic of China.

APPENDIX IX

Syllabus in Mathematics for Classes IX and X for the Central Board of Secondary Education

Paper I	$2\frac{1}{2}$ hours	75 Marks
Algebra—Unit I, VII, VIII, X, XI		—40 Marks
Commercial Mathematics—Unit II		—25 Marks
Mensuration—Unit III		—10 Marks
Paper II	$2\frac{1}{2}$ hours	75 Marks
Geometry	Unit IV	—35 Marks
Trigonometry	Unit V	—10 Marks
Matrices	Unit IX	—10 Marks
Statistics	Unit VI	—20 Marks

No question will be set in the examination in respect of the items contained in the last two units.

It is expected that the concepts and notations introduced in unit I will be utilized as and when considered appropriate while dealing with the rest of the syllabus.

In respect of the last two units it is expected that the teacher during his teaching will give relevant historical references and will also emphasise the concepts mentioned in the last unit on the 'Nature of Mathematical Thinking'.

Unit I—Revision 32 Periods

1.1 Set language and set notation.

1.2 The number system : Natural numbers, integers, fractions, rational numbers, irrational numbers, decimal fractions.

1.3 Relations and functions : The composite of functions, one-one functions.

Unit II—Commercial Mathematics 48 Periods

2.1 Compound interest, compound interest tables, investments and loans in banks, hire-purchase problems, payments in equal instalments.

2.2 Realistic problems of elementary nature dealing with rates and taxes.

2.3 Problems of percentages, profit and loss, shares and discount.

2.4 Use of logarithmic tables in solving problems of commercial mathematics.

Unit III—Mensuration 24 Periods

3.1 Areas of triangles, parallelograms and circles and areas of those figures which can be split up into these figures.

3.2 Surface area and volume of cuboid, lateral surface and volume of right pyramid, right prism, right circular cone and cylinder. Surface area and volume of sphere.

3.3 Finding areas of two-dimensional irregular figures.

Unit IV—Geometry 104 Periods

Of the geometrical results given below, some results have been set in bold type. It is expected that the truth of the results will be brought home to the students through measurements, observations, etc without any attempt to deduce them from the other results. The candidates will, however, be expected to deduce the results set in bold type. Riders may be asked pertaining to the results stated below both assumed/deduced.

It is expected that while dealing with the results, the students may reformulate the statements making use of the concepts of reflection, translation and rotation, etc.

4.1 Angles at a point

(*i*) If a ray stands on a line, the sum of the two adjacent angles so formed is two right angles; and the converse.

(*ii*) If two lines intersect, the vertically opposite angles are equal.

4.2 Parallel lines

(*i*) When a transversal intersects two parallel lines, then

(*a*) corresponding angles are equal,

(*b*) alternate angles are equal,

(*c*) sum of the inerior angles on the same side of the transversal is 180°; and the converse.

(*ii*) Lines which are parallel to the same line are parallel to each other.

4.3 Triangles and rectilinear figures

(*i*) **The sum of the three angles of a triangle is 180°.**

(*ii*) If two angles and the included side of one triangle are equal to the corresponding angles and the included side of the other triangle, the two triangles are congruent.

(*iii*) If two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of the other triangle, the two triangles are congruent.

(*iv*) If three sides of one triangle are equal to the corresponding sides of the other triangle, the two triangles are congruent.

(*v*) If two right triangles have their hypotenuses equal and one side of the one equal to on side of the other, the two triangles are congruent.

(*vi*) **If two sides of a triangle are equal, the angles opposite to these sides are equal ; and the converse.**

(*vii*) If two sides of a triangle are unequal, the longer side has the greater angle opposite to it ; and the converse.

(*viii*) Of all the segments that can be drawn to a given line from a given point outside it, the perpendicular segment is the shortest.

(*ix*) **The opposite sides and angles of a parallelogram are equal ; each diagonal bisects the parallelogram and the diagonals bisect each other.**

(*x*) **If a pair of opposite sides of a quadrilateral are equal and parallel, it is a parallelogram.**

(*xi*) The line, drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.

(*xii*) The segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

(*xiii*) If there are three or more parallel lines and the intercepts made by them on a transversal are equal, the corresponding intercepts on any other transversal are also equal.

4.4 Loci

(*i*) The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the segment joining the two points.

(*ii*) The locus of a point which is equidistant from two intersecting lines consists of the pair of lines which bisect the angles between the two given lines.

4.5 Areas

(*i*) **Parallelograms on the same base and between the same parallels are equal in area**

(*ii*) Triangles on the same or equal bases and of the same or equal altitudes are equal in area.

(*iii*) Triangles with equal areas and on the same or equal bases are of same or equal altitudes.

4.6 Similar triangles

(Proofs which are applicable only to commensurable magnitudes will be accepted.)

(*i*) If a line is drawn parallel to one side of a triangle, the other two sides are divided in the same ratio ; and the converse.

(*ii*) **If two triangles are equiangular, their corresponding sides are in the same ratio ; and the converse.**

(*iii*) **If an angle of one triangle is equal to the corresponding angle of the other triangle and the sides including these angles are in the same ratio, the triangles are similar.**

(*iv*) **If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.**

(*v*) **The bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle ; and likewise the bisector of the exterior angle.**

(*vi*) **The ratio of the areas of similar triangles is equal to the ratio of the squares on the corresponding sides.**

(*vii*) **PYTHAGORAS THEOREM—In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides ; and the converse.**

4.7 **The Circle**

(*i*) **Perpendicular from the centre of a circle to a chord bisects the chord, and the converse.**

(*ii*) **There is one and only one circle, which passes through three given points not in a line.**

(*iii*) **Equal chords of a circle are equidistant from the centre ; and the converse**

(*iv*) **The tangent at any point of a circle and the radius through that point are perpendicular to each other.**

(*v*) **Lengths of the two tangents to a circle from an external point are equal.**

(*vi*) **If two circles touch each other, the point of contact lies on the line joining their centres**

(*vii*) **The angle subtended by an arc of a circle at its centre is double the angle subtended by it at any point on the remaining part of the circle.**

(*viii*) **Angles in the same segment of a circle are equal ; and if a linesegment joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.**

(*ix*) **The angle in a semi-circle is a right angle ; and the converse.**

(*x*) **The opposite angles of any quadrilateral inscribed in a circle are supplementry, and the converse**

(*xi*) In congruent circles (or in the same circle)

(*i*) If two arcs subtend equal angles at their centres, they are congruent.

(*ii*) If two arcs are congruent, they subtend equal angles at their centres

(*xii*) In congruent circles (or in the same circle), if two chords are equal, they cut off congruent arcs ; conversely, if two arcs are congruent, the chords of the arcs are equal.

(*xiii*) **If a line touches a circle and if from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.**

(*xiv*) **If two chords of a circle intersect, either inside or outside the circle, the rectangle contained by the parts of one is equal in area to the rectangle contained by the parts of the other.**

Unit V—Trigonometry **24 Periods**

5.1 Sine x, Cosine x, Tangent x when $0 \leqslant x \leqslant 90°$, Tan 90° being not defined.

5.2 Values of Sin x, Cos x and Tan x, for $x = 0°, 30°, 45°, 60°$ and $90°$, Tan 90° being not defined.

5.3 Use of trigonometrical tables.

5.4 Simple cases of heights and distances.

Unit VI—Statistics **32 Periods**

6.1 Role of statistics in everyday life.

6.2 Collection and tabulation of statistical data.

6.3 Graphical representation of statistical data. Frequency polygons, Histograms, Bar Charts, Pie Charts, etc.

6.4 Measures of central tendency and dispersion, calculations are required only for Mean and Standard Deviation.

Unit VII—Systems of Equations and Inequations and their Graphical Representation **24 Periods**

7.1 Graphical representation in Cartesian plane of first degree equations and inequations in one or two variables.

7.2 Graphical illustration to show that the two linear equations in two unknowns may have no solution, one solution, or an infinity of solutions.

7.3 Solution of two simultaneous linear equations in two unknowns and the graphical interpretation of their solutions

7.4 Practical problems leading to two simultaneous linear equations or inequations in two variables, and their solutions

Unit VIII—Linear Programming 16 Periods

Linear programming · Problems and their formulation in two variables with illustrations from various fields and their solutions by graphical representation

Unit IX—Matrices 16 Periods

Matrices as stores of information, Route matrices, Net-work matrices, Addition of matrices

Note. Matrices with rows and columns not exceeding three will only be considered

Unit X—Quadratic Polynomials 16 Periods

10.1 Graphical representation of a quadratic polynomial in one variable.

10.2 Factors of Quadratic Polynomial ax^2+bx+c, $a, b, c \in R$. Solution set of $ax^2+bx+c \leqslant 0$ through factorization and their graphical representation.

Unit XI—Permutations, Combinations and the Binomial Theorem 24 Periods

11.1 Values of ${}^nC_r$, ${}^nP_r$ for given numerical values of n and r; $n \leqslant 5$ (the general formulae for ${}^nC_r$ and ${}^nP_r$ will not be asked)

11.2 Expansion of $(x+a)^n$ as a polynomial for a given value of $n \leqslant 5$ (proof of Binomial Theorem is not required.)

Unit XII—History of Mathematics with Special Reference to India

Brief History of Mathematics with special reference to contributions made by India, illustration of the contributions of Aryabhata, Bhaskara and Brahamgupta.

Unit XIII—Nature of Mathematical Thinking

13.1 Recognition of patterns of numbers and geometrical forms.

13.2 Generalization from particular cases and recognition that generalisation is not sufficient unless it is established mathematically.

13.3 Formulation of problems in terms of mathematical symbols and analysis of problems.

13.4 Discussion of the Polya's method 'How to Solve it'.

13.5 Discussion of what we mean by necessary and sufficient conditions.

13.6 Discussion of the axiomatic method.

13.7 Role of inductive and deductive reasoning in Mathematics.

13.8 Discussion of the concepts of precision, consistency and depth in Mathematics.

13.9 Formulation of problems in everyday life in mathematical forms and their solutions.

ANSWERS

CHAPTER XVIII

Exercise 18.1

1. 8 fans, 12 sewing machines
2. $P = 16$
3. $C = 0$
4. 4 units of A, 4 units of B ; Rs 40.00
5. A : 5 days, B : 3 days
6. 30 packages of screws 'A' and 20 packages of screws 'B' ; Rs 41.00
7. 4 pedestal lamps, 4 wooden shades
8. Number of quintals to be transported

		From	
		A	B
	D	10	50
To	E	50	0
	F	40	0

9. Delivery of petrol

		From (in litres)	
		A	B
	D	500	4000
To	E	3000	0
	F	3500	0

CHAPTER XIX

Exercise 19.1

1. $\begin{pmatrix} 240 & 360 \\ 180 & 420 \end{pmatrix}$; 2×2
2. $\begin{pmatrix} 2.00 \\ 1.50 \end{pmatrix}$; 2×1
3. (*i*) $M = \begin{pmatrix} 31 & 29 & 33 \\ 6 & 8 & 4 \end{pmatrix}$

   (*a*) 2, 3

   (*b*) Number of students who passed in each of the three subjects, number of students who passed/failed in chemistry,

   (*c*) Number of students who failed in chemistry, number of students who passed in physics

   (*ii*) $N = \begin{pmatrix} 31 & 6 \\ 29 & 8 \\ 33 & 4 \end{pmatrix}$
4. (*i*) 3×3, (*ii*) 2×2, (*iii*) 1×2, (*iv*) 4×1, (*v*) 2×2, (*vi*) 2×2.

Exercise 19.2

1. $$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Exercise 19.3

1. (i)
$$\begin{array}{c} \\ AB \\ AD \\ BC \\ CD \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

(ii)
$$\begin{array}{c} \\ AB \\ AD \\ BD \\ BC \\ CD \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

(iii)
$$\begin{array}{c} \\ AB \\ BC \\ CD \\ DE \\ EF \\ FA \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Exercise 19.4

1. (a) School A $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

School B $\begin{pmatrix} 7 & 5 \\ 6 & 8 \end{pmatrix}$

(b) Yes ; they have the same order

(c) $\begin{pmatrix} 11 & 8 \\ 12 & 10 \end{pmatrix}$

2. (a) $\begin{pmatrix} 3 & \underset{\cdots}{1} \\ \underset{\cdots}{2} & \underset{\cdots}{0} \end{pmatrix}$

(b) $\begin{pmatrix} 6 & \underset{\cdots}{7} & \underset{\cdots}{9} \\ \underset{\cdot\,\cdot}{-3} & \underset{\cdots}{1} & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & \underset{\cdots}{2} \\ \underset{\cdots}{1} & 1 \end{pmatrix} + \begin{pmatrix} \underset{\cdots}{3} & 2 \\ 0 & \underset{\cdots}{-1} \end{pmatrix}$

3. (a) $\begin{pmatrix} 9 & 8 \\ 0 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -3 & 3 & -1 \\ -3 & -4 & -2 \\ -2 & 2 & 18 \end{pmatrix}$

(c) $\begin{pmatrix} 14 & -7 & 0 & 13 \\ 12 & 5 & 20 & -6 \\ 11 & 9 & 2 & 9 \end{pmatrix}$

(d) $\begin{pmatrix} 1+a & b \\ c & d \end{pmatrix}$

4. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

5 $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

6. (a) Yes ; $\begin{pmatrix} 3 & -5 \\ -8 & 11 \end{pmatrix}$

(b) Yes ; $\begin{pmatrix} 3 & -5 \\ -8 & 11 \end{pmatrix}$

7. $\begin{pmatrix} 12 & 2 \\ 6 & 1 \\ -10 & 2 \end{pmatrix}$

8. (a) $5\begin{pmatrix} \underset{\cdots}{1} & \underset{\cdots}{0} \\ \underset{\cdots}{0} & \underset{\cdots}{1} \end{pmatrix}$

(b) $\begin{pmatrix} 12 & \underset{\cdots}{18} & -6 \\ \underset{\cdots}{36} & 12 & \underset{\cdots}{0} \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ \frac{2}{3} & \frac{4}{3} & 4 \\ 0 & -\frac{1}{3} & 3 \end{pmatrix}$

(d) $-\frac{2}{3}\begin{pmatrix} 8 & -\frac{3}{2} & 0 \\ 3 & -\frac{9}{2} & 6 \\ -9 & -12 & -12 \end{pmatrix}$

(e) $-4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$

9. $p = -2, q = 1$

10. (a) $\begin{pmatrix} 3 & 5 \\ 6 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 5 \\ 6 & -5 \end{pmatrix}$

Exercise 19.5

1. (a) $\begin{pmatrix} -22 & 16 \\ 5 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 21 \\ 2 & 14 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}$

5. (20)

6. $\begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$

7. (a) $\begin{pmatrix} -8 & 0 \\ 15 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} -8 & 0 \\ 15 & 0 \end{pmatrix}$

8. (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} \cos^2\theta - \sin^2\theta & 0 \\ 2\sin\theta\cos\theta & 1 \end{pmatrix}$

Exercise 19 6

1. (a) $\begin{pmatrix} 2 & 5 \\ 4 & -3 \end{pmatrix}$, $\begin{pmatrix} 11 \\ 9 \end{pmatrix}$

(b) $x = 3, y = 1$

2. $x = 4, y = -1$

MISCELLANEOUS EXERCISE
(On Chapter XIX)

1. $\begin{pmatrix} 391064 & 414406 \\ 75798 & 94199 \\ 27477 & 38488 \end{pmatrix}$

2. $\begin{pmatrix} 25.16 & 22.63 \\ 18.23 & 19.09 \\ 25.35 & 31.51 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

4. $\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} AC \\ AE \\ BD \\ BF \\ DF \\ CE \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$

5. (a) $\begin{pmatrix} -5 & 19 \\ 17 & 16 \\ 2 & -16 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 11 & 9 & -14 & 1 \\ 6 & 1 & 3 & 7 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} -3 & -5 \\ -10 & -20 \\ -\frac{2}{7} & -\frac{1}{7} \end{pmatrix}$

(d) $\begin{pmatrix} 13 & 4 \\ 24 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} -36 & 1 \\ 3 & -3 \end{pmatrix}$

(f) $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$

6. $x = -2$

7. (a) $\begin{pmatrix} 3 & 8 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ 31 \end{pmatrix}$

(b) $\begin{pmatrix} 18 & 25 \\ 2 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ 9 \end{pmatrix}$

8. (a) $x = 2, y = 3$

(b) $x = 1, y = 4$

9. $AB = \begin{pmatrix} -26 & 4 & 5 \\ 14 & -3 & -6 \\ 2 & 5 & 5 \end{pmatrix}$

$BA = \begin{pmatrix} -18 & 5 & 12 \\ -8 & 1 & 4 \\ 16 & 13 & -7 \end{pmatrix}$

CHAPTER XX

Exercise 20.2

1. $(x-6)(x+6)$
2. $\left[x+\left(\frac{3-\sqrt{65}}{4}\right)\right]\left[-x-\left(\frac{3+\sqrt{65}}{4}\right)\right]$
3. $(7x+4)(x+1)$
4. $\frac{1}{2}(x-2)(x-4)$
5. $(7x+1)(2x+1)$
6. $(x-1)(2x-1)$
7. $(11bx+2c)(bx+c)$
8. Not factorizable over R
9. $(2x-5)(2x-3)$
10. $(rx-p)(px+2q)$
11. $(x-2\sqrt{2})(\sqrt{2}x+1)$
12. Not factorizable over R
13. $(px-3q)(x+4p)$
14. $(3ax-b)(ax+b)$
15. Not factorizable over R
16. $(4x-\sqrt{3})(\sqrt{3}x+2)$
17. Not factorizable over R
18. (a) For all k, such that $k \geqslant 2$ or $k \leqslant -2$
    (b) For all k, such that $-2 < k < 2$

Exercise 20·3

1. (i) $\frac{1}{2}, \frac{1}{2}$ (ii) $-\frac{10}{9}, -\frac{10}{9}$
   (iii) $\frac{3}{2}, -\frac{3}{2}$ (iv) $-\frac{3}{5}, -\frac{3}{5}$
   (v) $\frac{1}{3}, \frac{3}{2}$ (vi) $\frac{q}{ap}, -\frac{q}{ap}$

(*vii*) $-\frac{bc}{ad}, -\frac{bc}{ad}$

(*viii*) 3, 4 (*ix*) $\frac{a+b}{5}, \frac{a-b}{2}$

(*x*) $10\sqrt{3}, -10\sqrt{3}$

(*xi*) $-\frac{1}{rp}, -\frac{1}{rp}$

2. (*i*) No real roots
 (*ii*) Irrational and unequal
 (*iii*) Rational and unequal
 (*iv*) Rational and unequal
 (*v*) Rational and equal

3. (*i*) $\frac{11}{6}, \frac{1}{2}$ (*ii*) $0, -\frac{1}{4}$

 (*iii*) $-\frac{11}{\sqrt{3}}, 6$

4. (*i*) $2x^2-5x+2=0$
 (*ii*) $x^2-3=0$ (*iii*) $x^2-x-12=0$

Exercise 20.4

1. -4
2. $\frac{1}{2}, \frac{4}{3}$
3. $-2, -\frac{3}{2}$
4. 4, 1 is extraneous
5. 2, 5
6. $\frac{6}{5}$, 5 is extraneous
7. $\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}$
8. $\frac{-7\pm\sqrt{17}}{2}, \frac{-7\pm\sqrt{33}}{2}$
9. $\sqrt{2}, -\sqrt{2}$,
10. 62, 63 or $-62, -63$
11. $\frac{2}{3}, \frac{3}{2}$
12. 5, 12
13. Length = 10 m, breadth = 7 m
14. 12, 15 or $-15, -12$
15. 5%
16. 12 cubits
17. 9, 42

Exercise 20.5

1. (*i*) $x \geqslant \frac{2}{5}$ or $x \leqslant \frac{-2}{3}$
 (*ii*) $x < 1$ or $x > 6$
 (*iii*) $x > 2$ or $x < -2$
 (*iv*) $-3 \leqslant x \leqslant 3$
 (*v*) All real x

2. (*i*) No solution (*ii*) $x \geqslant 7$ or $x \leqslant -3$
 (*iii*) $-2 \leqslant x \leqslant \frac{1}{2}$ (*iv*) $-4 \leqslant x \leqslant 3$

MISCELLANEOUS EXERCISE
(On Chapter XX)

2. (*i*) $(\sqrt{3}\,y+4)(y+2\sqrt{3})$
 (*ii*) $(10x+3)(x+15)$
 (*iii*) Not factorizable over R
 (*iv*) $(3x+4)(-5x+9)$
 (*v*) Not factorizable over R

3. (*i*) $\frac{5}{4}, -1$ (*ii*) $\frac{5}{2}, \frac{8}{3}$

 (*iii*) $\frac{b}{a}, -\frac{2b}{3a}$ (*iv*) $\frac{1}{3}, -\frac{1}{16}$

 (*v*) $\frac{12}{7}, -\frac{3}{4}$ (*vi*) $-2, -\frac{4a}{3}$

 (*vii*) $-\frac{3t}{rs}, -\frac{3t}{rs}$

 (*viii*) $-\frac{1}{a^2}, \frac{1}{b^2}$

 (*ix*) $\frac{13}{\sqrt{7}}, -\sqrt{7}$ (*x*) $\frac{b}{a}, \frac{2b}{a}$

4. (*i*) Irrational and unequal
 (*ii*) No real roots
 (*iii*) Rational and unequal
 (*iv*) Equal roots

5. (*i*) $-2, 4$ (*ii*) $-\frac{10}{\sqrt{3}}, \frac{7}{a}$

 (*iii*) $\frac{17}{28}, -2$ (*iv*) $2a, a^2$

6 (*i*) $6x^2-5x+1=0$
(*ii*) $x^2-12=0$
(*iii*) $x^2+ax-2a^2=0$
(*iv*) $4x^2-16x+9=0$

7. (*i*) $k=4$
(*ii*) $-\frac{8}{5}<k<\frac{8}{5}$
(*iii*) $k=2$ or $k=-\frac{10}{9}$

8. (*i*) No real roots (*ii*) 5, -8
(*iii*) $\frac{1}{4}, \frac{1}{13}$
(*iv*) $-6+\sqrt{6}, -6-\sqrt{6}, -6+\sqrt{31}, -6-\sqrt{31}$
(*v*) 4, $\frac{13}{2}$
(*vi*) No real roots, -1 is an extraneous root

9. 16 or 48

10. $\frac{15}{4}$ cubits

11. 3, $\frac{1}{3}$

13. (*i*) All real values of x
(*ii*) $-4 \leqslant x \leqslant 5$
(*iii*) No solution in reals

CHAPTER XXI

Exercise 21.1

1. 6
2. (*a*) 25, (*b*) 20
3. (*a*) 125, (*b*) 60
4. 6
5. 6
6. 24
7. 12
8. 120
9. 12
10. $2^3, 2^4, 2^5$; Yes

Exercise 21.2

1. 6
2. (*a*) 20, (*b*) 16
3. 1728
5. 12
6. 6720
7. (*i*) 3, (*ii*) 2
8. (*i*) 5, (*ii*) 4
10. 120

Exercise 21.3

1. 7200 3. 10
4. 10 5. 10
6. 2880 7. 3
9. 5 10. 40

Exercise 21.4

1. $x^6+3x^4y^2+3x^2y^4+y^6$
2. $x^2+2+\frac{1}{x^2}$
3. $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$
4. $\frac{1}{16}x^8-\frac{1}{8}x^6y+\frac{3}{32}x^4y^2-\frac{1}{32}x^2y^3+\frac{1}{256}y^4$
5. $-27x^3-9x-\frac{1}{x}-\frac{1}{27x^3}$
6. 6
7. $\frac{5}{256}y^4$
8. $\frac{3}{2}x^2y^6$

MISCELLANEOUS EXERCISE

(On Chapter XXI)

1. 6, 6^2, 6^3; Yes 2. (*a*) 25, (*b*) 20
3. (*i*) 85, (*ii*) 155 4. 120, 6
5. 325, 65 6. 6
7. 12 8. 5040
9. 120 10. 120
11. 10 12. 3, 6, 10
13. 6, 10, 15 14. 20, 6
15. 2^5 16. (*b*) $2^3, 2^4, 2^n$
17. 30 18. 30
19. 10 21. 5

22. (a) $a^3-6a^2b+12ab^2-8b^3$

(b) $\frac{16}{81}t^4-\frac{16}{9}t^2+6-9\frac{1}{t^2}+\frac{81}{16}\frac{1}{t^4}$

(c) $1024x^5-6400x^4y+16000x^3y^2$ $-20000x^2y^3+12500xy^4-3125y^5$

(d) $y^6+9y^4x+27y^2x^2+27x^3$

23. $-90a^2b^3$

24 $-96x^3$

25. $10x^6a^4$ and $10x^4a^6$

CHAPTER XXII

Exercise 22.1

2. 24 cm
3. 5 cm
4. (i) 1 cm, (ii) 7 cm
11. Diameter perpendicular to the chord

Exercise 22.2

13. A circle whose centre is the same as the centre of the given circle and radius is the perpendicular distance from the centre to the chord

Exercise 22 3

1. 80°

CHAPTER XXIII

Exercise 23.1

1. 12 cm 7. 4 cm

Exercise 23 2

1. (i) 30°, (ii) 60°, (iii) 90°, (iv) 90°, (v) 30°

Exercise 23.3

1. (i) 3 cm (ii) 5 cm (iii) 10 cm or 3 cm
2. (i) 11 cm (ii) 2 cm
3. 6 cm

CHAPTER XXIV

Exercise 24.1

1. 60 2. 4 : 1
3. 49 m 4. 14 cm
5. 9 m 6. 4. 6 cm
7 71.04 sq cm 8. 4 cm, $4\sqrt{3}$ cm
9. 6 cm, 12 cm
10. 24 cm, 10 cm, 120 sq. cm
11. 14 m, 10 m

Exercise 24.2

1. 6550 sq. m 2. $\frac{3\sqrt{3}}{2}b^2$ sq. m
3. 3409 sq. m

Exercise 24 3

1 Rs 4 19 2. 57.75 sq. cm
3. 66 cm 4. 1980 m
5. 9.46 cm, 4.07 sq. cm
6. Rs 1386

Exercise 24.4

1. 600 2. $\frac{1}{4}$ m
3. 432 sq. cm 4. 1440 sq cm
5. 109.3 sq. m, 6. Rs 281.25

Exercise 24.5

1. 126 sq. cm, 50.85 cu. cm
2. Rs 20
3. 1215 cu cm
4. Rs 84.84, 7.3177, cu. m
5. 146.25 cu. m

Exercise 24.6

1. Rs 233.14
2. Rs 31.40
3. .019625 cu. m
4. cylindrical tin, 1475.9 cu. cm
5. 175 cu. cm
6. 2.5 m
7. Rs 475.20

Exercise 24.7

1. 260 sq. cm, 400 cu cm

2. Rs 987 24
3. 9735 sq. m
4 13 m, 5 m
5. 360 cu. m, $108\sqrt{5}$ sq. m
6. 314 cu. cm, 204.1 sq. cm

Exercise 24 8

1 1386 sq cm, 4851 cu. cm
2 226.08 sq cm, 452 16 cu cm
3 103 62 sq. cm
4. 21 cm, 5544 sq cm
5. Yes
6 30.48 cu. m
7. 2.1 cm
8 512
9. 16.01 cm
10 36 m

INDEX